OKAN ÜNİVERSİTESİ
MÜHENDİSLİK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2017–05–23, 9:00-11:00 **MATH115 Basic Mathematics – Final Exam** N. Course

FORENAME:

SURNAME:

STUDENT NO:

SIGNATURE:

Time Allowed: **120** min.

Answer **4** questions.

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have **120** minutes to answer **4** questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You must answer in English.
4. You must show your working for all questions.
5. This exam contains 8 pages. Check to see if any pages are missing.
6. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the first 20 minutes, or in the final 10 minutes, of the exam.
7. Switch your mobile phone off and seal it in the envelope provided. Do not open your envelope until the exam is finished or you have left the room.
8. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
9. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
10. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam **120** dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan **4** tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce veriniz.
4. Sonuca ulaşmak için yaptığınız işlemleri ayrıntılarıyla gösteriniz.
5. Sınav 8 sayfadan oluşmaktadır. Lütfen eksik sayfa olup olmadığını kontrol edin.
6. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın ilk 20 dakikası ve son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
7. Cep telefonunuzu kapatınız ve size verilen zarfın içine koyunuz. Zarfı, sınav süresi bitene kadar ya da sınav salonundan çıkana kadar açmayınız.
8. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
9. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
10. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	4	5	TOTAL
25	25	25	25	25	100

Question 1 (Integration)

(a) [5 pts] Calculate $\displaystyle\int_0^4 \left(3x - \frac{x^3}{4}\right)\, dx$.

An easy question to begin with.

$$\begin{aligned}\int_0^4 \left(3x - \frac{x^3}{4}\right)\, dx &= \left[\frac{3}{2}x^2 - \frac{x^4}{16}\right]_0^4 \\ &= \left(\frac{3}{2}(4)^2 - \frac{(4)^4}{16}\right) - (0-0) \\ &= 24 - 16 = 8.\end{aligned}$$

(b) [10 pts] Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_0^{7x} \frac{1}{1+t^2}\, dt$.

By the Fundamental Theorem of Calculus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_0^{7x} \frac{1}{1+t^2}\, dt \\ &= 7\frac{d}{du}\int_0^{u} \frac{1}{1+t^2}\, dt \\ &= 7\frac{1}{1+u^2} \\ &= \frac{7}{1+49x^2}.\end{aligned}$$

(c) [10 pts] Calculate $\displaystyle\int \sec^2(3x+2)\, dx$.

Let $u = 3x+2$. Then $du = 3\ dx$. Since $\frac{d}{du}\tan u = \sec^2 u$, we have that

$$\begin{aligned}\int \sec^2(3x+2)\, dx &= \int \frac{1}{3}\sec^2 u\ du \\ &= \frac{1}{3}\tan u + C \\ &= \frac{1}{3}\tan(3x+2) + C.\end{aligned}$$

Question 2 (The Chain Rule)

(a) [1 pts] Please write your student number at the top-right of this page.

Suppose that $f(u) = \frac{2u}{u^2+1}$ and $u = g(x) = 2x^2 + x + 1$.

(b) [24 pts] Calculate $(f \circ g)'(0)$ and $(f \circ g)'(-1)$.

By the Chain Rule,

$$(f \circ g)'(x) = f'\big(g(x)\big)g'(x).$$

Clearly $g'(x) = 4x + 1$. Moreover

$$\begin{aligned} f'(u) = \frac{d}{du}\left(\frac{2u}{u^2+1}\right) &= \frac{(2u)'(u^2+1) - (2u)(u^2+1)'}{(u^2+1)^2} \\ &= \frac{2(u^2+1) - (2u)(2u)}{(u^2+1)^2} = \frac{2-2u^2}{(u^2+1)^2}. \end{aligned}$$

Now $g(0) = 1$, $g(-1) = 2$, $g'(0) = 1$ and $g'(-1) = -3$. Since

$$f'(1) = \frac{2-2}{(1+1)^2} = 0$$

and

$$f'(2) = \frac{2-8}{(4+1)^2} = -\frac{6}{25},$$

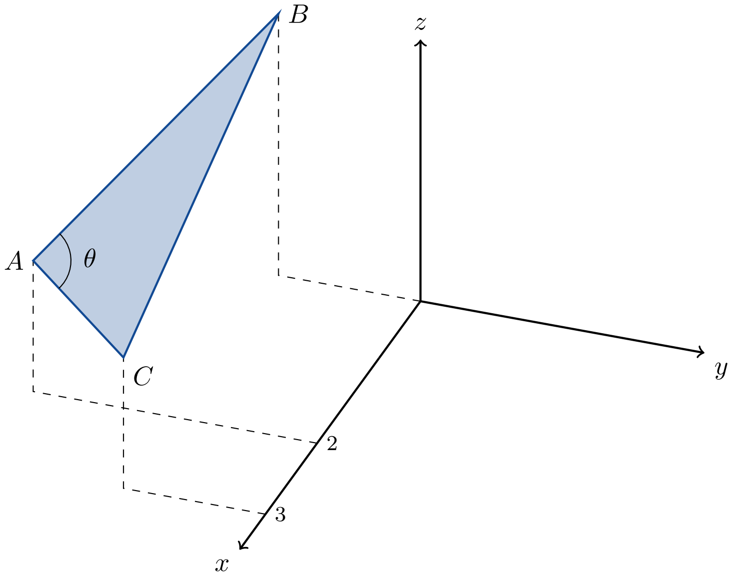
we have that

$$(f \circ g)'(0) = f'\big(g(0)\big)g'(0) = f'(1)g'(0) = 0$$

and

$$(f \circ g)'(-1) = f'\big(g(-1)\big)g'(-1) = f'(2)g'(-1) = -\frac{6}{25} \times -3 = \frac{18}{25}.$$

Question 3 (Triangles) Consider the triangle with vertices at $A(2,-2,1)$, $B(0,-1,2)$ and $C(3,-1,1)$.



(a) [3 pts] Find $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

We have that

$$\begin{aligned}\overrightarrow{AB} &= B - A = (0,-1,2) - (2,-2,1) = (-2,1,1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k},\\ \overrightarrow{AC} &= C - A = (3,-1,1) - (2,-2,1) = (1,1,0) = \mathbf{i} + \mathbf{j}\\ \overrightarrow{BC} &= C - B = (3,-1,1) - (0,-1,2) = (3,0,-1) = 3\mathbf{i} - \mathbf{k}.\end{aligned}$$

(b) [7 pts] Find $\cos\theta$.

We can calculate that

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \frac{(-2)(1) + (1)(1) + (1)(0)}{\sqrt{(-2)^2 + 1^2 + 1^2}\sqrt{1^2 + 1^2}} = \frac{-1}{\sqrt{6}\sqrt{2}} = \frac{-1}{2\sqrt{3}}.$$

(question 3 continued)

(c) [15 pts] Find the area of the triangle with vertices at A, B and C.

Note that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\ &= \big((1)(0) - (1)(1)\big)\mathbf{i} - \big((-2)(0) - (1)(1)\big)\mathbf{j} + \big((-2)(1) - (1)(1)\big)\mathbf{k} \\ &= -\mathbf{i} + \mathbf{j} - 3\mathbf{k}.\end{aligned}$$

The area of the traiangle is

$$\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\left\|-\mathbf{i} + \mathbf{j} - 3\mathbf{k}\right\| = \frac{1}{2}\sqrt{(-1)^2 + 1^2 + (-3)^2} = \frac{1}{2}\sqrt{1+1+9} = \frac{\sqrt{11}}{2}.$$

Question 4 (Cylindrical and Spherical Polar Coordinates)

(a) [7 pts] Convert the cylindrical polar coordinates $(r, \theta, z) = (1, 45°, 1)$ into spherical polar coordinates.

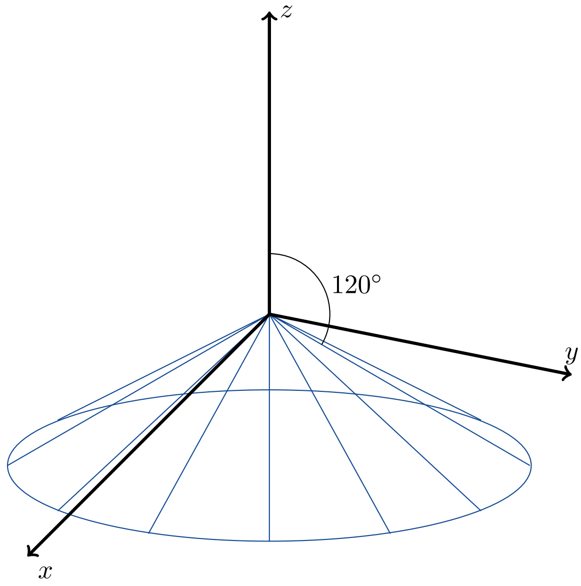
First we calculate that $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} = \sqrt{1+1} = \sqrt{2}$. Then $\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{2}}$ which implies that $\phi = 45°$. Thus

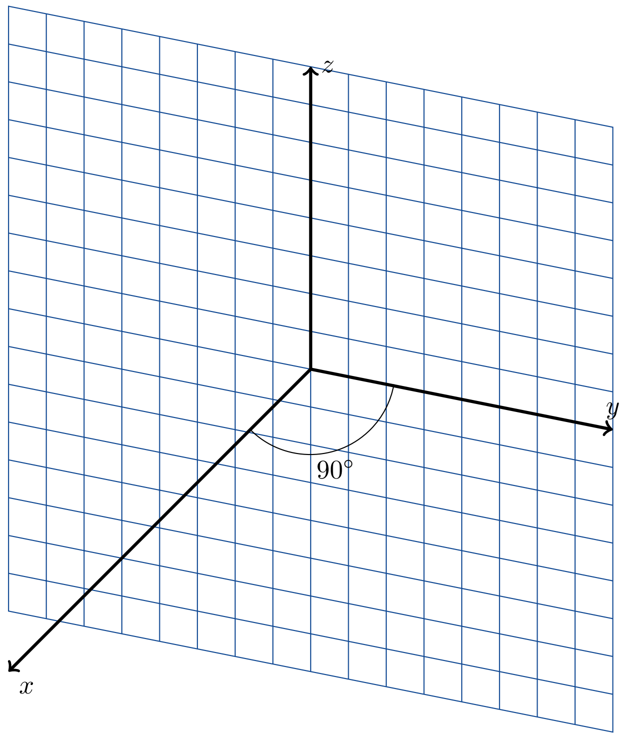
$$(\rho, \theta, \phi) = (\sqrt{2}, 45°, 45°).$$

(question 4 continued)

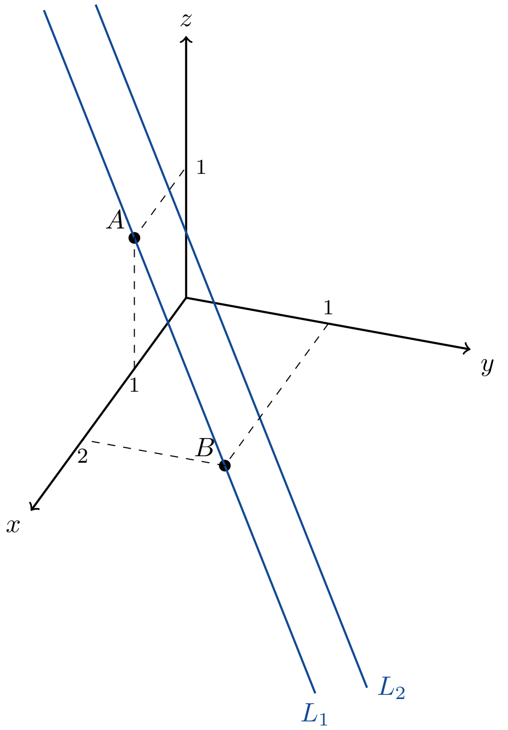
(b) [9 pts] Sketch the surface $\phi = 120°$ in $\mathbb{R}^3$.

(c) [9 pts] Sketch the surface $\theta = 90°$ in $\mathbb{R}^3$.





Question 5 (Lines) Let L_1 be the line which passes through the points $A(1,0,1)$ and $B(2,1,0)$. Let L_2 be the line $x=-1-3t$, $y=-1-3t$, $z=1.5+3t$.



(a) [5 pts] Find parametric equations for L_1.

Let $\mathbf{v}_1 = \overrightarrow{AB} = B - A = (2,1,0)-(1,0,1) = (1,1,-1) = \mathbf{i}+\mathbf{j}-\mathbf{k}$. Using $P_0 = A(1,0,1)$, we have $x = 1+t$, $y = t$, $z = 1-t$.

(b) [5 pts] Show that L_1 is parallel to L_2.

Clearly $\mathbf{v}_2 = -3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is parallel to L_2. Since $v_2 = -3v_1$, the two lines are parallel.

(c) [15 pts] Find the distance between L_1 and L_2.

The distance between the two lines is equal to the distance from A to L_2. Let $P = P(-1,-1,1.5)$ be a point on L_2 and let $S = A(1,0,1)$.
We calculate that $\overrightarrow{PS} = S - P = (1,0,1) - (-1,-1,1.5) = (2,1,-0.5)$ and

$$\begin{aligned}\overrightarrow{PS} \times \mathbf{v}_2 &= (2,1,-0.5)\times(-3,-3,3)\\ &= \big((1)(3)-(-0.5)(-3)\big)\mathbf{i}\\ &\quad -\big((2)(3)-(-0.5)(-3)\big)\mathbf{j}\\ &\quad +\big((2)(-3)-(1)(-3)\big)\mathbf{k}\\ &= 1.5\mathbf{i} - 4.5\mathbf{j} - 3\mathbf{k}.\end{aligned}$$

Then the distance from S to L_2 is

$$\begin{aligned} d &= \frac{\left\|\overrightarrow{PS}\times\mathbf{v}_2\right\|}{\|\mathbf{v}_2\|}\\ &= \frac{\sqrt{(1.5)^2+(-4.5)^2+(-3)^2}}{\sqrt{(-3)^2+(-3)^2+3^2}}\\ &= \frac{\sqrt{\frac{9}{4}+\frac{27}{4}+\frac{36}{4}}}{3\sqrt{3}}\\ &= \frac{\sqrt{18}}{3\sqrt{3}} = \frac{3\sqrt{2}}{3\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}.\end{aligned}$$

$$\cos\theta = \sin\left(\tfrac{\pi}{2} - \theta\right)$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$
$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\cos 0 = \cos 0^\circ = 1$$
$$\cos\tfrac{\pi}{6} = \cos 30^\circ = \tfrac{\sqrt{3}}{2}$$
$$\cos\tfrac{\pi}{4} = \cos 45^\circ = \tfrac{1}{\sqrt{2}}$$
$$\cos\tfrac{\pi}{3} = \cos 60^\circ = \tfrac{1}{2}$$
$$\cos\tfrac{\pi}{2} = \cos 90^\circ = 0$$

$$\sin 0 = \sin 0^\circ = 0$$
$$\sin\tfrac{\pi}{6} = \sin 30^\circ = \tfrac{1}{2}$$
$$\sin\tfrac{\pi}{4} = \sin 45^\circ = \tfrac{1}{\sqrt{2}}$$
$$\sin\tfrac{\pi}{3} = \sin 60^\circ = \tfrac{\sqrt{3}}{2}$$
$$\sin\tfrac{\pi}{2} = \sin 90^\circ = 1$$

$$(uv)' = uv' + u'v$$
$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$
$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$

$$\tfrac{d}{dx}x^n = nx^{n-1}$$
$$\tfrac{d}{dx}\sin x = \cos x$$
$$\tfrac{d}{dx}\cos x = -\sin x$$
$$\tan x = \tfrac{\sin x}{\cos x} \qquad \tfrac{d}{dx}\tan x = \sec^2 x$$
$$\sec x = \tfrac{1}{\cos x} \qquad \tfrac{d}{dx}\sec x = \sec x\tan x$$
$$\cot x = \tfrac{\cos x}{\sin x} \qquad \tfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$
$$\operatorname{cosec} x = \tfrac{1}{\sin x} \qquad \tfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x\cot x$$
$$\tfrac{d}{dx}e^x = e^x$$
$$\tfrac{d}{dx}\ln|x| = \tfrac{1}{x}$$

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$V = \int_a^b A(x)\,dx \qquad V = \int_a^b \pi\big(R(x)\big)^2\,dx$$

$$c = \sqrt{a^2-b^2} \quad \text{or} \quad c = \sqrt{a^2+b^2}$$

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \qquad \theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$
$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$
$$d = \frac{\left\|\overrightarrow{PS}\times\mathbf{v}\right\|}{\|\mathbf{v}\|} \qquad d = \frac{\left|\overrightarrow{PS}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$$
$$d = \frac{\left\|\overrightarrow{P_1P_2}\times\mathbf{v}_1\right\|}{\|\mathbf{v}_1\|} \qquad d = \frac{\left|\overrightarrow{P_1P_2}\cdot(\mathbf{v}_1\times\mathbf{v}_2)\right|}{\|\mathbf{v}_1\times\mathbf{v}_2\|}$$