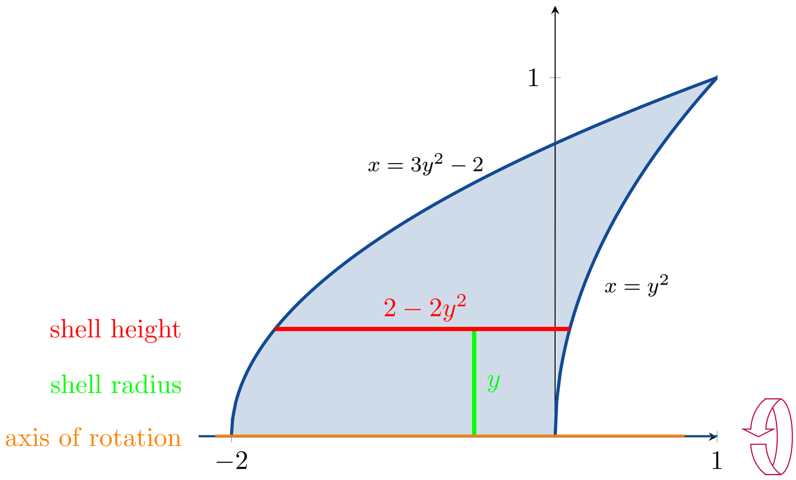
2017.01.03 **MATH115 Basic Mathematics – Final Exam Solutions** N. Course

Question 1 (The Shell Method). [25 pts] The region bounded by $x = 3y^2 - 2$, $x = y^2$ and $y = 0$ (for $y \geq 0$) is shown below. This region is rotated about the x-axis to generate a solid. **Use the shell method** to find its volume.



The shell radius (show in green above) is y. The shell height (show in red above) is $y^2 - (3y^2 - 2) = 2 - 2y^2$.
Since $3y^2 - 2 = x = y^2 \implies y = \pm 1$, we must integrate from $y = 0$ to $y = 1$. Therefore the volume of the solid is

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dy = \int_0^1 2\pi(y)(2 - 2y^2) \, dy$$
$$= \pi \int_0^1 4y - 4y^3 \, dy = \pi \left[2y^2 - y^4\right]_0^1 = \pi.$$

Question 2 (Linear Systems). [25 pts] Use Gauss-Jordan elimination to solve

$$\begin{aligned} 2x_1 - x_2 + 4x_3 + 4x_4 &= 21 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ -5x_2 + 5x_3 + 5x_4 &= 35 \\ -x_1 + 3x_2 &= -28. \end{aligned}$$

An augmented matrix for this linear system is

$$\begin{bmatrix} 2 & -1 & 4 & 4 & 21 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & -5 & 5 & 5 & 35 \\ -1 & 3 & 0 & 0 & -28 \end{bmatrix}$$

To solve the linear system, we must row reduce this matrix:

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 4 & 4 & 21 \\ 0 & -5 & 5 & 5 & 35 \\ -1 & 3 & 0 & 0 & -28 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$ and $R_4 + R_1 \to R_4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -3 & 2 & 2 & 21 \\ 0 & -5 & 5 & 5 & 35 \\ 0 & 4 & 1 & 1 & -28 \end{bmatrix}$$

$-\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & -7 \\ 0 & -5 & 5 & 5 & 35 \\ 0 & 4 & 1 & 1 & -28 \end{bmatrix}$$

$R_3 + 5R_2 \to R_3$ and $R_4 - 4R_2 \to R_4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & -7 \\ 0 & 0 & \frac{5}{3} & \frac{5}{3} & 0 \\ 0 & 0 & \frac{11}{3} & \frac{11}{3} & 0 \end{bmatrix}$$

$\frac{3}{5}R_3 \to R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & -7 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{11}{3} & \frac{11}{3} & 0 \end{bmatrix}$$

$R_4 - \frac{11}{3}R_3 \to R_4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & -7 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 + \frac{2}{3}R_3 \to R_2$ and $R_1 - R_3 \to R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 - R_2 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

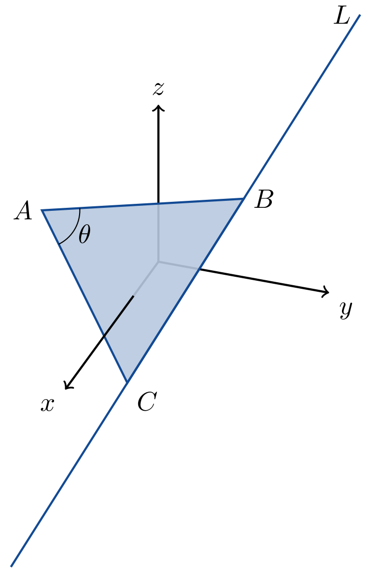
Changing back into linear equations, we have that $x_1 = 7, x_2 = -7$, $x_3 + x_4 = 0$ and $0 = 0$. Therefore

$$\begin{aligned} x_1 &= 7 \\ x_2 &= -7 \\ x_3 &= t \\ x_4 &= -t \end{aligned}$$

for all $t \in \mathbb{R}$.

Question 3 (Geometry). Consider the triangle with vertices at $A(1,-1,1)$, $B(0,1,1)$ and $C(1,0,-1)$.

(a) [3 pts] Find $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$.



We have that

$$\begin{aligned}
\overrightarrow{AB} &= B - A = (0,1,1) - (1,-1,1) = (-1,2,0) = -\mathbf{i} + 2\mathbf{j},\\
\overrightarrow{AC} &= C - A = (1,0,-1) - (1,-1,1) = (0,1,-2) = \mathbf{j} - 2\mathbf{k}\\
\overrightarrow{BC} &= C - B = (1,0,-1) - (0,1,1) = (1,-1,-2) = \mathbf{i} - \mathbf{j} - 2\mathbf{k}.
\end{aligned}$$

(b) [5 pts] Find $\cos\theta$.

We can calculate that

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \frac{(-1)(0)+(2)(1)+(0)(-2)}{\sqrt{(-1)^2+2^2+0^2}\sqrt{0^2+1^2+(-2)^2}} = \frac{2}{\sqrt{5}\sqrt{5}} = \frac{2}{5}.$$

(c) [5 pts] Find parametric equations for the line L passing through B and C.

Let $\mathbf{v} = \overrightarrow{BC} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The parametric equations for the line passing through B in the direction $\mathbf{v}$ are

$$x = t, \qquad y = 1 - t, \qquad z = 1 - 2t.$$

(d) [12 pts] Find the area of the triangle with vertices at A, B and C.

Note that

$$\begin{aligned}
\overrightarrow{AB}\times\overrightarrow{AC} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\\
&= \big((2)(-2) - (0)(1)\big)\mathbf{i} - \big((-1)(-2) - (0)(0)\big)\mathbf{j} + \big((-1)(1) - (2)(0)\big)\mathbf{k}\\
&= -4\mathbf{i} - 2\mathbf{j} - \mathbf{k}.
\end{aligned}$$

The area of the parallelogram determined by A, B and C is

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \|-4\mathbf{i} + 2\mathbf{j} - \mathbf{k}\| = \sqrt{(-4)^2 + 2^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}.$$

Since the triangle is half of the parallelogram, the area of the triangle is $\frac{\sqrt{21}}{2}$.

Question 4 (Differentiation).

(a) [6 pts] Find $f'(3) = \left.\frac{df}{dx}\right|_{x=3}$ if $f(x) = \displaystyle\int_0^x \left(t^2+1\right)^6 \; dt$.

Clearly

$$f'(x) = \frac{d}{dx}\int_0^x \left(t^2+1\right)^6 \; dt = \left(x^2+1\right)^6$$

by the Fundamental Theorem of Calculus. Therefore

$$f'(3) = \left(3^2+1\right)^6 = 10^6 = 1000000.$$

(b) [12 pts] Find the (natural) domain and the critical points of $y = x\sqrt{4-x^2}$.

The domain of this function is $[-2,2]$ since $\sqrt{4-x^2}$ is undefined if $x^2 > 4$. $\boxed{2}$

Note that

$$\frac{d}{dx}\sqrt{4-x^2} = \frac{d}{dx}\left(4-x^2\right)^{\frac{1}{2}} = -2x\frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}} = -\frac{x}{\sqrt{4-x^2}}$$

by the chain rule.
By the product rule (using $u = x$ and $v = \sqrt{4-x^2}$) we calculate that

$$\frac{dy}{dx} = u'v + uv' = \frac{d}{dx}(x)\sqrt{4-x^2} + x\frac{d}{dx}\left(\sqrt{4-x^2}\right) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}. \qquad \boxed{4}$$

Clearly $\frac{dy}{dx}$ does not exist if $x = \pm 2$. $\boxed{2}$ Solving

$$\begin{aligned} 0 = \frac{dy}{dx} &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ \sqrt{4-x^2} &= \frac{x^2}{\sqrt{4-x^2}} \\ 4-x^2 &= x^2 \\ 4 &= 2x^2 \end{aligned}$$

gives $x = \pm\sqrt{2}$. $\boxed{2}$
Therefore the critical points of $y = x\sqrt{4-x^2}$ are -2, $-\sqrt{2}$, $\sqrt{2}$ and 2. $\boxed{2}$

Recall that $\cot x = \frac{\cos x}{\sin x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.

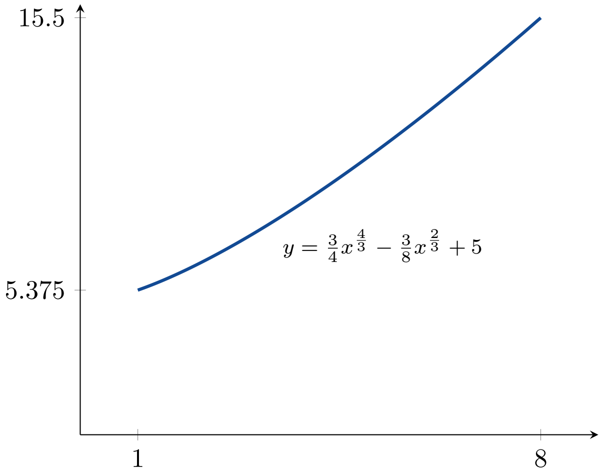
(c) [7 pts] Use the quotient rule to show that $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$.

By the quotient rule, we have that

$$\begin{aligned} \frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \left(\frac{u}{v}\right)' &= \frac{u'v - uv'}{v^2} \\ &= \frac{(\cos x)'\sin x - (\sin x)'\cos x}{\sin^2 x} = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x \end{aligned}$$

since $\sin^2 x + \cos^2 x = 1$.

Question 5 (Arc Length). [25 pts] Find the length of the curve $y = \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{8}x^{\frac{2}{3}} + 5$ from $x = 1$ to $x = 8$.



Since

$$\frac{dy}{dx} = x^{\frac{1}{3}} - \frac{1}{4}x^{-\frac{1}{3}},$$

we have that

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(x^{\frac{1}{3}} - \frac{1}{4}x^{-\frac{1}{3}}\right)^2 = 1+x^{\frac{2}{3}}-\frac{1}{2}+\frac{1}{16}x^{-\frac{2}{3}} = x^{\frac{2}{3}}+\frac{1}{2}+\frac{1}{16}x^{-\frac{2}{3}} = \left(x^{\frac{1}{3}} + \frac{1}{4}x^{-\frac{1}{3}}\right)^2.$$

Therefore the length of the curve is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^8 x^{\frac{1}{3}} + \frac{1}{4}x^{-\frac{1}{3}}\; dx \\ &= \left[\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{8}x^{\frac{2}{3}}\right]_1^8 = \left(\frac{3}{4}(16) + \frac{3}{8}(4)\right) - \left(\frac{3}{4} + \frac{3}{8}\right) \\ &= \left(12 + \frac{12}{8}\right) - \frac{9}{8} = \frac{96}{8} + \frac{12}{8} - \frac{9}{8} = \frac{99}{8}. \end{aligned}$$