Question 3 (Differentiation and Extreme Values of Functions)

(a)-(c) Find the derivatives below. In each part, you must state which differentiation rules or other theorems you are using.

(a) [10 pts] Find $\frac{dg}{dx}$ if $g(x) = \frac{x^2-4}{x+0.5}$.

(8 points for correct answer with correct working. 2 points for stating which rules/theorems were used.)

Using the quotient rule (and sum and difference rules) we have that

$$\begin{aligned}\frac{dg}{dx} &= \frac{d}{dx}\left(\frac{x^2-4}{x+0.5}\right) \\ &= \frac{(x^2-4)'(x+0.5)-(x^2-4)(x+0.5)'}{(x+0.5)^2} \\ &= \frac{2x(x+0.5)-(x^2-4)(1)}{(x+0.5)^2} \\ &= \frac{2x^2+x-x^2+4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}.\end{aligned}$$

(b) [10 pts] Find $\frac{dy}{dt}$ if $y = \sin(t^2+t-1)$.

(As above.)

Let $u = t^2+t-1$. Using the chain rule, we calculate that

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du}\frac{du}{dt} = (\cos u)(2t+1) \\ &= (2t+1)\cos(t^2+t-1).\end{aligned}$$

(c) [10 pts] Find $\frac{d^2r}{d\theta^2}$ if $r = \theta^3\cos\theta$.

(As above.)

Using the product rule, we have that

$$\frac{dr}{d\theta} = (\theta^3)'\cos\theta + \theta^3(\cos\theta)' = 3\theta^2\cos\theta - \theta^3\sin\theta.$$

Differentiating a second time, again using the product rule, gives

$$\begin{aligned}\frac{d^2r}{d\theta^2} &= \frac{d}{d\theta}\left(3\theta^2\cos\theta - \theta^3\sin\theta\right) \\ &= (3\theta^2)'\cos\theta + 3\theta^2(\cos\theta)' - (\theta^3)'\sin\theta - \theta^3(\sin\theta)' \\ &= 6\theta\cos\theta - 3\theta^2\sin\theta - 3\theta^2\sin\theta - \theta^3\cos\theta \\ &= 6\theta\cos\theta - 6\theta^2\sin\theta - \theta^3\cos\theta.\end{aligned}$$

Define a function $h : [-1, 8] \to \mathbb{R}$ by $h(x) = \sqrt[3]{x}$.

(d) [20 pts] Find the absolute maximum and absolute minimum values of h on $[-1, 8]$.

(5 points for finding h'. 5 points for finding the critical point. 10 points for finding abs. max. and abs. min.)

The derivative of $h(x) = x^{\frac{1}{3}}$ is $h'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. h' does not exist if $x = 0$. h' is never equal to zero. Therefore the only critical point of h is $x = 0$.

We calculate that $h(-1) = -1$, $h(0) = 0$ and $h(8) = 2$. Therefore the absolute maximum value of h on $[-1, 8]$ is 2, and the absolute maximum value of h on $[-1, 8]$ is -1.

OKAN ÜNİVERSİTESİ
MÜHENDİSLİK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2016.11.15 | **MATH115 Basic Mathematics – Midterm Exam** | N. Course

FORENAME: ÖRNEKTİR

SURNAME: SAMPLE

STUDENT NO:

SIGNATURE:

Time Allowed: **60** min.

Answer **2** questions.

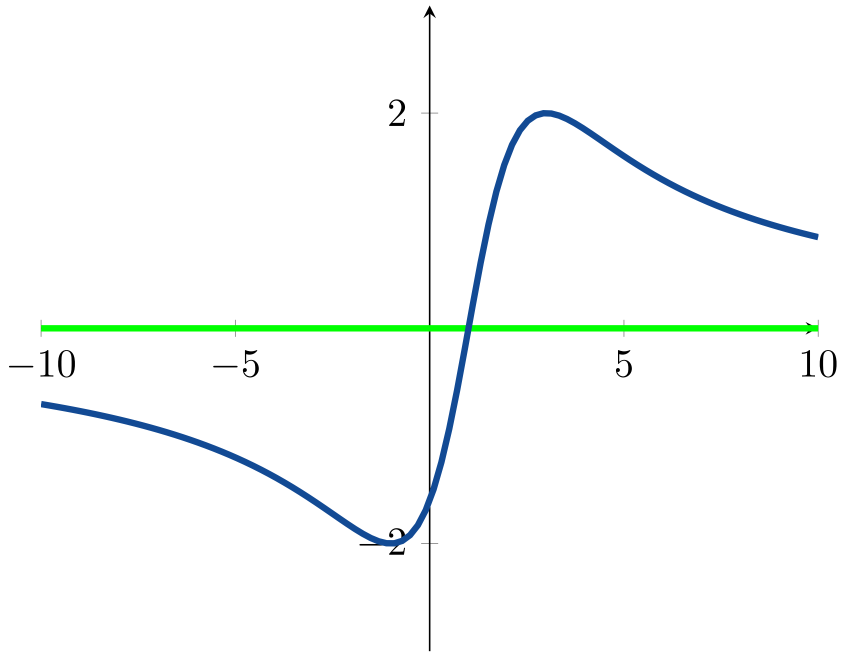
Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have **60** minutes to answer **2** questions from a choice of 3. If you choose to answer more than 2 questions, then only your best 2 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You must answer in English.
4. You must show your working for all questions.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the first 20 minutes, or in the final 10 minutes, of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam **60** dakikadır. Sınavda 3 soru sorulmuştur. Bu sorulardan **2** tanesini seçerek cevaplayınız. 2'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 2 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce veriniz.
4. Sonuca ulaşmak için yaptığınız işlemleri ayrıntılarıyla gösteriniz.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın ilk 20 dakikası ve son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	TOTAL
/50	/50	/50	/100

Question 1 (Concavity and Curve Sketching) Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{8x-8}{x^2-2x+5}$.



(a) [4 pts] Find the horizontal asymptote(s) of $y = f(x)$. Draw the asymptote(s) on the axes above.

Since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\infty} \frac{8x-8}{x^2-2x+5} = \lim_{x\to\infty} \frac{\frac{8}{x}-\frac{8}{x^2}}{1-\frac{2}{x}+\frac{5}{x^2}} = \frac{0-0}{1-0+0} = 0,$$

the line $y = 0$ is the only horizontal asymptote of $= f(x)$.

(b) [3 pts] The derivative of f is given by $f'(x) = \frac{-8(x-3)(x+1)}{(x^2-2x+5)^2}$. Find all the critical points of f.

Clearly $f'(x) = 0$ if and only if $x = -1$ or $x = 3$. Since f' exists for all x, the only critical points of f are $x = -1$ and $x = 3$.

(c) [2 pts] Calculate $f(1)$ and $f'(1)$.

We can calculate that

$$f(1) = \frac{8-8}{1-2+5} = 0$$

and that

$$f'(1) = \frac{-8(-2)(2)}{(1-2+5)^2} = \frac{32}{4^2} = 2.$$

(d) [5 pts] Complete the following table:

interval	$(-\infty,-1)$	$(-1,3)$	$(3,\infty)$
sign of f'	$f' < 0$	$f' > 0$	$f' < 0$
behaviour of f	decreasing	increasing	decreasing

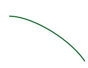
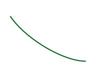
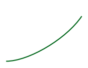
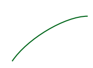
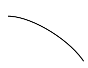
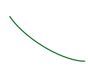
(e) [4 pts] The second derivative of f is given by $f''(x) = \frac{16(x-1)(x^2-2x-11)}{(x^2-2x+5)^3}$. Solve $f''(x) = 0$.

Clearly $f''(1) = 0$. 1Since the roots of $x^2 - 2x - 11 = 0$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4+44}}{2} = 1 \pm 2\sqrt{3}$, we can see that $f''(x) = 0$ if and only if $x = 1-2\sqrt{3}$ or 1 or $1+2\sqrt{3}$.

(f) [6 pts] Complete the following table:

interval	$(-\infty, 1-2\sqrt{3})$	$(1-2\sqrt{3}, 0)$	$(1, 1+2\sqrt{3})$	$(1+2\sqrt{3}, \infty)$
sign of f''	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
concavity of f	concave down	concave up	concave down	concave up

(g) [6 pts] Complete the following table:

$(-\infty, 1-2\sqrt{3})$	$(1-2\sqrt{3}, -1)$	(-1,1)	$(1,3)$	$(3, 1+2\sqrt{3})$	$(1+2\sqrt{3}, \infty)$
decreasing and concave down	decreasing and concave up	increasing and concave up	increasing and concave down	decreasing and concave down	decreasing and concave up
					

(h) [20 pts] Draw the graph of $y = f(x)$ on the axes above.
[HINT: $\sqrt{3} \approx 1.7$]

Question 2 (Limits and Continuity)

(a)-(b) Calculate the following limits. For each one, you must state which limit laws or theorems you are using.

(a) [12 pts] $\lim_{x\to 0} \dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x} =$

(8 points for correct answer. 4 points for stating which laws/theorems were used.)

Using the sum, difference and quotient rules, we can calculate that

$$\lim_{x\to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x\to 0} \frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim_{x\to 0} \frac{2x}{x(x-1)(x+1)} = \lim_{x\to 0} \frac{2}{x^2-1} = \frac{2}{0-1} = -2$$

(b) [12 pts] $\lim_{x\to \pi} \dfrac{x-1+\sin x}{3\cos x} =$

(As above.)

Using the sum, difference and quotient rules, we can calculate that $\lim_{x\to\pi} \dfrac{x-1+\sin x}{3\cos x} = \dfrac{\pi-1+0}{3(-1)} = \dfrac{1-\pi}{3}$.

(c) [13 pts] The inequality

$$1-\frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

holds for values of x close to zero. What, if anything, does this tell you about $\lim_{x\to 0} \dfrac{x\sin x}{2-2\cos x}$? State which limit laws or other theorems you are using.

(8 points for correct value, 3 points for mentioning the Sandwich Theorem and 2 points for mentioning the other rules.)

First,

$$\lim_{x\to 0} 1-\frac{x^2}{6} = \lim_{x\to 0} 1 - \lim_{x\to 0} \frac{x^2}{6} = 1-0 = 1$$

(difference, constant multiple and power rules) and $\lim_{x\to 0} 1 = 1$. By the Sandwich Theorem, we have that $\lim_{x\to 0} \frac{x\sin x}{2-2\cos x} = 1$.

Define a function $g:\mathbb{R}\to\mathbb{R}$ by $g(x) = \begin{cases} \frac{x^2-x-6}{x-3} & x \neq \pm 3, \\ -5 & x=-3, \\ 5 & x=3. \end{cases}$

(d) [13 pts] At which points is g continuous? Justify your answer.

Since $\lim_{x\to -3} f(x) = \frac{(-3)^2-(-3)-6}{(-3)-3} = \frac{9+3-6}{-6} = -1$ and $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = x+2 = 5$, we have that f is continuous at $x=3$, but discontinuous at $x=-3$.

Since f is a rational function $\frac{p(x)}{q(x)}$ and since $q(x)\neq 0$ if $x \neq 3$, we can see that f is continuous at all $x \neq \pm 3$ also.

Therefore f is discontinuous at $x=-3$ and continuous everywhere else.

(9 points for correctly proving that f is continuous at 3, but discontinuous at -3. Remaining 4 points for considering $x \neq \pm 3$.)