Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

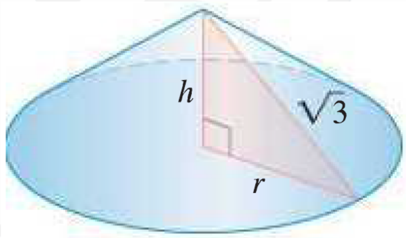
Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

Problem	Points	Score
1	17	
2	15	
3	15	
4	20	
5	17	
6	16	
Total:	100	

Do not write in the table to the right.

1. (a) 9 Points A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the *radius, height, and volume* of the cone of greatest volume that can be made this way.



Solution: Assuming the two legs of the right triangle is h and r, then $h^2+r^2 = 3 \Rightarrow r = \sqrt{3-h^2}$ (pythogorean rule) the volume of the cone is $1/3\pi r^2 h = 1/3\pi h\cdot(3-h^2) = 1/3\pi(3h-h^3)$ for $0<h<\sqrt{3}$ take its first derivative and set it to 0 $h^2=1$, the critical point occurs at $h=1$ (height) $y=\sqrt{3-1} = \sqrt{2} = 1.414$ (radius). Notice that $dV/dh > 0$ for $0<h<1$ and $dV/dh<0$ for $1<h<\sqrt{3}$. Therefore this critical point corresponds to a maximum. The cone of greatest volume has radius $\sqrt{2}$m, height 1m and volume $\frac{2\pi}{3}$m^3.

p.222, pr.27

(b) 8 Points Find the *slope* of the curve $y=y(x)$ defined by $x^3y^3+y^2=x+y$ at the point $(1,1)$.

Solution: We differentiate the given equation implicitly.

$$\frac{d}{dx}\left(x^3y^3+y^2\right) = \frac{d}{dx}(x+y)$$
$$3x^2y^3+3x^3y^2y'+2yy' = 1+y'$$
$$(3x^3y^2+2y-1)y' = 1-3x^2y^3$$
$$\Rightarrow y' = \frac{1-3x^2y^3}{3x^3y^2+2y-1}$$

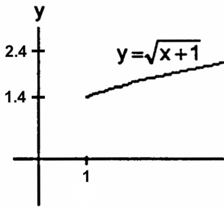
Hence the slope is

$$m = y'\big|_{(x,y)=(1,1)} = \frac{1-3}{3+2-1} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$$

p.177, pr.87

2. The region bounded by the curve $y = \sqrt{x+1}$, x-axis and the lines $x = 1$ and $x = 5$ is revolved about x-axis.

   (a) $\boxed{\text{8 Points}}$ Find the *area of this surface*.



Solution:

$$y = \sqrt{x+1} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x+1)} \Rightarrow S = \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx$$

$$\begin{aligned}
S &= 2\pi\int_1^5 \sqrt{(x+1)+\frac{1}{4}}\,dx \\
&= 2\pi\int_1^5 \sqrt{x+\frac{5}{4}}\,dx = 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5 = \frac{4\pi}{3}\left[\left(5+\frac{5}{4}\right)^{3/2} - \left(1+\frac{5}{4}\right)^{3/2}\right] \\
&= \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left(\frac{5^3}{2^3} - \frac{3^3}{2^3}\right) \\
&\longrightarrow S = \frac{\pi}{6}(125-27) = \frac{98\pi}{6} = \boxed{\frac{49\pi}{3}}
\end{aligned}$$

p.335, pr.16

   (b) $\boxed{\text{7 Points}}$ Find the *volume* of this solid.

Solution: Here we slice vertically which makes x the choice for the intrgration varibale. When revolved about x-axis, the region generates a solid of revolution and the slice generates a disk, a thin coin-shaped object. Note that $\Delta V \approx \pi[R(x)]^2\Delta x$. Note also that $R(x) = \sqrt{x+1}$ and so $\Delta V \approx \pi[\sqrt{x+1}]^2\Delta x = \pi(x+1)\Delta x$. The volume of the solid is

$$\begin{aligned}
V &= \int_1^5 \pi(x+1)\,\mathrm{d}x = \pi\left[\frac{x^2}{2}+x\right]_1^5 \\
&= \pi\left[\frac{5^2}{2}+5-\left(\frac{1^2}{2}+1\right)\right] \\
&= \boxed{16\pi}
\end{aligned}$$

Alternatively, we can use the shells. We must slice horizontally. This leads to two different shells and two separate integrals.

$$\begin{aligned}
V &= \int_0^{1.4} 2\pi y(5-1)\,\mathrm{d}y + \int_{1.4}^{2.4} 2\pi y(5-(y^2-1))\,\mathrm{d}y \\
&= 8\pi\int_0^{1.4} y\,\mathrm{d}y + 2\pi\int_{1.4}^{2.4} y(6-y^2)\,\mathrm{d}y = 8\pi\left[\frac{y^2}{2}\right]_0^{1.4} + 2\pi\left[3y^2 - \frac{y^4}{4}\right]_{1.4}^{2.4} \\
&= 8\pi\frac{1.96}{2} + 2\pi\left(6(2.4) - \frac{(2.4)^3}{3} - \left(6(1.4) - \frac{(1.4)^3}{3}\right)\right)
\end{aligned}$$

p.73, pr.38

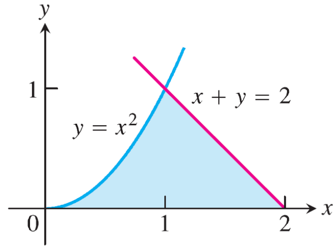
3. (a) $\boxed{\text{7 Points}}$ Find dy/dx if $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$ for $|x| < \pi/2$.

Solution: Let $u = \sin x$. Then since $|x| < \pi/2$, we have $|\cos x| = \cos x$ and so by the Fundamental Theorem of Calculus,

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^{\sin x}\frac{dt}{\sqrt{1-t^2}} = \frac{d}{dx}\int_0^u \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\cdot\cos x = \frac{\cos x}{\sqrt{1-(\sin x)^2}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|} = \boxed{1}$$

p.282, pr.39

(b) $\boxed{\text{8 Points}}$ Find the total area of the shaded region



Solution: Let R denote the shaded region. We shall find the area of R in two different ways. One description of R is

$$R = \underbrace{\{(x,y) \in \mathbb{R}^2 | \quad 0 \le y \le x^2, \quad 0 \le x \le 1\}}_{R_1} \cup \underbrace{\{(x,y) \in \mathbb{R}^2 | \quad 1 \le y \le 2-x, \quad 1 \le x \le 2\}}_{R_2}$$

Notice that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$. Therefore $A(R) = A(R_1) + A(R_2)$. This shows that we will need two integrals.

$$\begin{aligned} A(R) = A(R_1) + A(R_2) &= \int_0^1 x^2\,\mathrm{d}x + \int_1^2 (2-x)\,\mathrm{d}x \\ &= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 \\ &= \frac{1}{3} + \left(2(2) - \frac{4}{2}\right) - \left(2(1) - \frac{1}{2}\right) = \frac{1}{3} + \frac{1}{2} = \boxed{\frac{5}{6}} \end{aligned}$$

Next, we will find the area by integrating with respect to y. Another description for R is

$$R = \{(x,y) \in \mathbb{R}^2 | \quad \sqrt{y} \le x \le 2-y, \quad 0 \le y \le 1\}$$

This will, of course, require a single integral (easier than the first way). Hence

$$\begin{aligned} A = A(R) &= \int_0^1 (2 - y - \sqrt{y})\,dy \\ &= \left[2y - \frac{y^2}{2} - \frac{y^{3/2}}{3/2}\right]_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \boxed{\frac{5}{6}} \end{aligned}$$

p.298, pr.36

4. Consider the function $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$. You may assume that $y' = -\dfrac{1}{(x-1)^2}$ and $y'' = \dfrac{2}{(x-1)^3}$. Use this information to graph the function.

(a) $\boxed{\text{4 Points}}$ Give the *domain* and *asymptotes*. Justify your answer.

For the domain, the answer is $\boxed{(-\infty, -1) \cup (-1, +1) \cup (1, +\infty)}$.

vertical asymptote(s)

Solution: First for the vertical asymptotes, we compute the four limits

$$\lim_{x\to 1^+} \frac{2x^2+x-1}{x^2-1} = \lim_{x\to 1^+} \frac{\cancel{(x+1)}(2x-1)}{\cancel{(x+1)}(x-1)} = \lim_{x\to 1^+} \frac{2x-1}{x-1} = +\infty$$

$$\text{and } \lim_{x\to 1^-} \frac{2x^2+x-1}{x^2-1} = \lim_{x\to 1^-} \frac{\cancel{(x+1)}(2x-1)}{\cancel{(x+1)}(x-1)} = \lim_{x\to 1^-} \frac{2x-1}{x-1} = -\infty$$

This shows that the line $\boxed{x=1}$ is a vertical asymptote. The next two limits are:

$$\lim_{x\to -1^+} \frac{2x^2+x-1}{x^2-1} = \lim_{x\to -1^+} \frac{\cancel{(x+1)}(2x-1)}{\cancel{(x+1)}(x-1)} = \lim_{x\to -1^+} \frac{2x-1}{x-1} = \frac{3}{2}$$

$$\text{and } \lim_{x\to -1^-} \frac{2x^2+x-1}{x^2-1} = \lim_{x\to -1^-} \frac{\cancel{(x+1)}(2x-1)}{\cancel{(x+1)}(x-1)} = \lim_{x\to -1^-} \frac{2x-1}{x-1} = \frac{3}{2}$$

This shows that the line cannot be a vertical asymptote. Hence the line $\boxed{x=1}$ is the only vertical asymptote for the graph.

p.212, pr.75

horizontal asymptote(s)

Solution: For the horizontal asymptotes, we need

$$\lim_{x\to\pm\infty} \frac{2x^2+x-1}{x^2-1} = \lim_{x\to\pm\infty} \frac{2x^2+x-1}{x^2-1}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\pm\infty} \frac{2+(1/x)-(1/x^2)}{1-(1/x^2)} = \frac{2+0-0}{1-0} = 2$$

and so the line $\boxed{y=2}$ is the horizontal asymptote.

p.212, pr.75

oblique asymptote(s)

Solution: No oblique asymptote can exist, as there is horizontal one.

p.212, pr.75

(b) $\boxed{\text{4 Points}}$ Find the intervals where the graph is increasing and decreasing. Find the local maximum and minimum values.

interval(s) where the graph is increasing:

Solution: If we examine the sign of $y' = -\dfrac{1}{(x-1)^2}$, we see that $y' < 0$ for all $x \neq 1$. This shows that the *graph is nowhere increasing*.

p.212, pr.75

interval(s) where the graph is decreasing:

Solution: If we examine the sign of $y' = -\dfrac{1}{(x-1)^2}$, we see that $y' < 0$ for all $x \neq 1$. This shows that the *graph is decreasing on the whole domain*.

p.212, pr.75

max/min value(s)

Solution: The graph has *no maximum and no minimum values*.

p.212, pr.75

(c) $\boxed{\text{4 Points}}$ Determine where the graph is concave up and concave down, and find any inflection points.

interval(s) of concave up:

Solution: If we examine the sign of $y'' = \dfrac{2}{(x-1)^3}$, we see that $y'' < 0$ if $x < 1$ and $y'' > 0$ if $x > 1$. Therfore the graph is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$. Although the concavity changes at $x = 1$, there is no point of inflection of this function.

p.212, pr.75

interval(s) of concave down:

Solution: If we examine the sign of $y'' = \dfrac{2}{(x-1)^3}$, we see that $y'' < 0$ if $x < 1$ and $y'' > 0$ if $x > 1$. Therfore the graph is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$. Although the concavity changes at $x = 1$, there is no point of inflection of this function.

p.212, pr.75

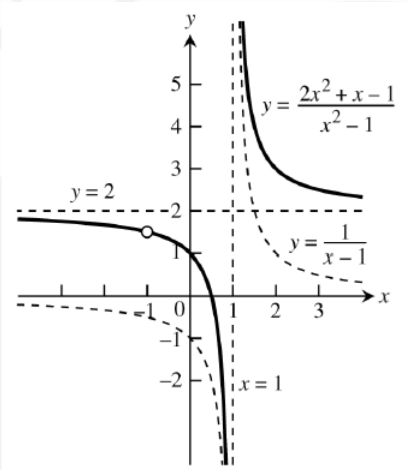
point(s) of inflection:

Solution: Although the concavity changes at $x = 1$, there is no point of inflection of this function.

p.212, pr.75

(d) $\boxed{\text{8 Points}}$ *Sketch a graph* of the function. Label the asymptotes, critical points and the inflection points.

Solution:



p.212, pr.75

5. (a) $\boxed{\text{9 Points}}$ $\displaystyle\lim_{x\to 0} \frac{8x}{3\sin x - x} = ?$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{8x}{3\sin x - x} &= \lim_{x\to 0} \frac{8x}{3\sin x - x} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to 0} \frac{8}{3\frac{\sin x}{x} - 1} = \frac{8}{3\underbrace{\left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\right]}_{=1} - 1} \\
&= \frac{8}{3(1) - 1} = \boxed{4}
\end{aligned}$$

p.97, pr.23

(b) $\boxed{\text{8 Points}}$ $y = 4x\sqrt{x + \sqrt{x}} \quad \Rightarrow \quad \dfrac{dy}{dx} = ?$

Solution: By the product rule, we have

$$\begin{aligned}
\frac{dy}{dx} &= 4x\frac{d}{dx}\left(\sqrt{x + \sqrt{x}}\right) + \left(\sqrt{x + \sqrt{x}}\right)\frac{d}{dx}(4x) \\
&= 4x\frac{1}{2\sqrt{x + \sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right) + 4\sqrt{x + \sqrt{x}}
\end{aligned}$$

p.176, pr.34

6. (a) $\boxed{\text{8 Points}}$ $\displaystyle\int x^{-1/3}(1 - x^{2/3})^{3/2}\, dx = ?$

Solution: Substitute $y = 1 - x^{2/3}$. Then $dy = -\frac{2}{3}x^{-1/3}\,\mathrm{d}x$. Hence

$$\begin{aligned}
\int x^{-1/3}(1-x^{2/3})^{3/2}\,\mathrm{d}x &= \frac{3}{2}\int (1-x^{2/3})^{3/2}\underbrace{\frac{2}{3}x^{-1/3}\,\mathrm{d}x}_{-dy} \\
&= -\frac{3}{2}\int y^{3/2}\,\mathrm{d}y \\
&= -\frac{3}{2}\left[\frac{y^{3/2+1}}{3/2+1}\right] + C \\
&= -\frac{3}{2}\cdot\frac{2}{5}y^{5/2} + C = \boxed{-\frac{3}{5}\left(1-x^{2/3}\right)^{5/2} + C}
\end{aligned}$$

p.302, pr.53

(b) $\boxed{\text{8 Points}}$ $\displaystyle\int_{\pi}^{3\pi} \cot^2\left(\frac{x}{6}\right)\mathrm{d}x = ?$

Solution: Let $u = \frac{x}{6}$. Then $du = \frac{1}{6}\mathrm{d}x$. When $x = \pi$, we have $u = \frac{\pi}{6}$ and when $x = 3\pi$, we have $u = \frac{\pi}{2}$. Therefore, by the trigonometric identity $\csc^2 u = \cot^2 u + 1$, we have

$$\begin{aligned}
\int_{\pi}^{3\pi}\cot^2\left(\frac{x}{6}\right)\mathrm{d}x = 6\int_{\pi}^{3\pi}\cot^2\left(\frac{x}{6}\right)\underbrace{\frac{1}{6}\,\mathrm{d}x}_{du} &= 6\int_{\pi/6}^{\pi/2}\cot^2 u\,\mathrm{d}u = 6\int_{\pi/6}^{\pi/2}(\csc^2 u - 1)\,\mathrm{d}u \\
&= 6\int_{\pi/6}^{\pi/2}\csc^2 u\,\mathrm{d}u - 6\int_{\pi/6}^{\pi/2}\mathrm{d}u \\
&= 6[-\cot u]_{\pi/6}^{\pi/2} - 6[u]_{\pi/6}^{\pi/2} \\
&= 6[-\cot\pi/2 + \cot\pi/6] - 6[\pi/2 - \pi/6] \\
&= 6(0+\sqrt{3}) - 3\pi + \pi = \boxed{6\sqrt{3} - 2\pi}
\end{aligned}$$

p.302, pr.59