Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

Do not write in the table to the right.

Problem	Points	Score
1	35	
2	34	
3	31	
Total:	100	

1. (a) $\boxed{\text{10 Points}}$ Find the derivative $\dfrac{dy}{dx}$ if $y = \ln(\cos^{-1} x)$.

Solution: By using Chain Rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\cos^{-1} x)\right) = \frac{1}{\cos^{-1} x}\frac{d}{dx}(\cos^{-1} x) = \frac{1}{\cos^{-1} x}\frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-1}{\sqrt{1-x^2}\cos^{-1} x}}$$

p.80, pr.23

(b) $\boxed{\text{12 Points}}$ Find the integral $\displaystyle\int \frac{1}{\sqrt{-2x-x^2}}\,\mathrm{d}x$.

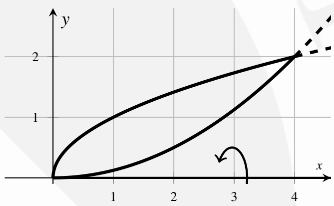
Solution: By completing the square, we have

$$\begin{aligned}\int \frac{1}{\sqrt{-2x-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-(x^2+2x+1)}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-(x+1)^2}}\,\mathrm{d}x\\ &= \int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\\ &= \sin^{-1} u + c = \boxed{\sin^{-1}(x+1)+c}\end{aligned}$$

where $u = x+1$ and $du = dx$.

p.115, pr.11

(c) $\boxed{\text{13 Points}}$ The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ is revolved about x-axis. Find the volume of the resulting solid.



Solution: If we use the shells, we slice horizontally and so we have

$$\begin{aligned}V &= \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}\,\mathrm{d}y\\ &= \int_0^1 2\pi y(\sqrt{8y}-y^2)\,\mathrm{d}y\\ &= 2\pi\int_0^0 (2\sqrt{2}y^{3/2}-y^3)\,\mathrm{d}y = 2\pi\left[\frac{4\sqrt{2}y^{5/2}}{5}-\frac{y^4}{4}\right]_0^2\\ &= 2\pi\left(\frac{(4)(2^3)}{5}-\frac{(4)(4)}{4}\right) = (2\pi)(4)\left(\frac{8}{5}-1\right) = \frac{8\pi}{5}(8-5) = \boxed{\frac{24\pi}{5}}.\end{aligned}$$

p.192, pr.87

2. (a) $\boxed{\text{12 Points}}$ Find the area of the surface generated by revolving the curve $x = \sqrt{2y-1}$, $5/8 \le y \le 1$ about y-axis.

Solution: We use the formula $\int_c^d 2\pi x\sqrt{1+(dx/dy)^2}\,\mathrm{d}y$ where $c = 5/8$, $d = 1$ and $x = \sqrt{2y-1}$. Hence

$$x = \sqrt{2y-1} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{2y-1}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1} \Rightarrow S = \int_{5/8}^1 2\pi\sqrt{2y-1}\sqrt{1+\frac{1}{2y-1}}\,dy$$

$$S = 2\pi\int_{5/8}^1 \sqrt{(2y-1)+1}\,dy = 2\pi\sqrt{2}\int_{5/8}^1 y^{1/2}\,dy = 2\pi\sqrt{2}\left[\frac{1}{3}y^{3/2}\right]_{5/8}^1$$

$$\longrightarrow S = \frac{4\pi\sqrt{2}}{3}\left[1^{3/2} - \left(\frac{5}{8}\right)^{3/2}\right] = \boxed{\frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right)}$$

p.191, pr.22

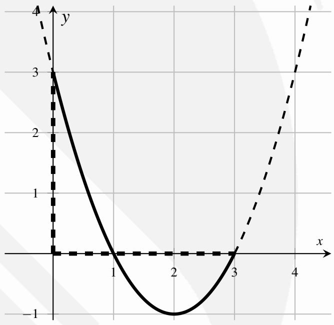
(b) $\boxed{\text{10 Points}}$ Use L'Hôpital's Rule to find the limit $\lim_{x\to\pi/2^-} \sec(7x)\cos(3x)$.

Solution: The limit leads to the indeterminate $\frac{0}{0}$. Hence using L'Hôpital's Rule, we have

$$\lim_{x\to\pi/2^-} \sec(7x)\cos(3x) = \lim_{x\to\pi/2^-} \frac{\cos(3x)}{\cos(7x)} \overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to\pi/2^-} \frac{-3\sin(3x)}{-7\sin(7x)} = \frac{-3\sin(3(\pi/2))}{-7\sin(7(\pi/2))} = \frac{(-3)(-1)}{(-7)(-1)} = \boxed{\frac{3}{7}}$$

p.94, pr.10

(c) $\boxed{\text{12 Points}}$ For $0 \le x \le 3$, find the total area between the curve $f(x) = x^2-4x+3$ and x-axis.



Solution: $x^2-4x+3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$;

$$\begin{aligned}\text{Area } &= \int_0^1 (x^2-4x+3)\,\mathrm{d}x - \int_1^3 (x^2-4x+3)\,\mathrm{d}x = \left[\frac{x^3}{3}-2x^2+3x\right]_0^1 - \left[\frac{x^3}{3}-2x^2+3x\right]_1^3 \\ &= \left[\frac{1^3}{3}-2(1)^2+3-0\right] - \left[\frac{3^3}{3}-2(3)^2+3(3)\right] + \left[\frac{1^3}{3}-2(1)^2+3(1)\right] \\ &= \left(\frac{1}{3}+1\right) - \left[0-\left(\frac{1}{3}+1\right)\right] = \boxed{\frac{8}{3}}\end{aligned}$$

p.192, pr.57

3. (a) $\boxed{\text{11 Points}}$ Evaluate the integral $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)\mathrm{d}t$.

Solution: Let $u = 1+\dfrac{1}{t} \Rightarrow \mathrm{d}u = -t^{-2}\,\mathrm{d}t$. When $t=-1$, we have $u=0$ and when $t=-1/2$, we have $u=-1$. Therefore

$$\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)\mathrm{d}t = \int_0^{-1} -\sin^2 u\,\mathrm{d}u = \int_0^{-1} -\frac{1}{2}(1-\cos(2u))\,\mathrm{d}u$$
$$= \left[-\left(\frac{u}{2}-\frac{1}{4}\sin(2u)\right)\right]_0^{-1}$$
$$= -\left[\left(\frac{1}{2}-\frac{1}{4}\sin(-2)\right)\right] - \left(\frac{0}{2}-\frac{1}{4}\sin(0)\right) = \boxed{\frac{1}{2}-\frac{1}{4}\sin(2)}$$

p.257, pr.4

(b) $\boxed{\text{10 Points}}$ For what value of m, if any, is $f(x) = \begin{cases}\sin(2x) & x\le 0\\ mx & x>0\end{cases}$ continuous at $x=0$? Differentiable at $x=0$? Justify your answer.

(a) For continuity at $x=0$:

Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\sin(2x) = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(mx) = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$, independent of m; since $f(0) = 0 = \lim_{x\to 0} f(x)$, it follows that f is continuous at $x=0$ for all values of m.

(b) For differentiability at $x=0$:

Solution: $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-}(\sin(2x))' = \lim_{x\to 0^-}(2\cos(2x)) = 2$ and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}(mx)' = \lim_{x\to 0^+} m = m \Rightarrow f$ is differentiable at $x=0$ provided that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \Rightarrow m=2$.

(c) $\boxed{\text{10 Points}}$ Use only optimization to find the point on the line $\dfrac{x}{a}+\dfrac{y}{b}=1$ *closest to the origin.*

Solution: Suppose $P(x,y)$ is the point on this line that is closest to the origin. Let $d=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$ and $\dfrac{x}{a}+\dfrac{y}{b}=1 \Rightarrow y = -\dfrac{b}{a}x+b$.
We can minimize d by minimizing $D = \left(\sqrt{x^2+y^2}\right)^2 = x^2+\left(-\dfrac{b}{a}x+b\right)^2 \Rightarrow D' = 2x+2\left(-\dfrac{b}{a}x+b\right)\left(-\dfrac{b}{a}\right) = 2x+\dfrac{2b^2}{a^2}x-\dfrac{2b^2}{a}$. Hence $D'=0 \Rightarrow 2\left(x+\dfrac{b^2}{a^2}x-\dfrac{b^2}{a}\right)=0 \Rightarrow x = \dfrac{ab^2}{a^2+b^2}$ is the critical point $\Rightarrow y = -\dfrac{b}{a}\left(\dfrac{ab^2}{a^2+b^2}\right)+b = \dfrac{a^2b}{a^2+b^2}$. Thus $D'' = 2+\dfrac{2b^2}{a^2} \Rightarrow D''\left(\dfrac{ab^2}{a^2+b^2}\right) = 2+\dfrac{2b^2}{a^2} > 0 \Rightarrow$ the critical point is local minimum by the Second Derivative Test, $\Rightarrow \boxed{\left(\dfrac{ab^2}{a^2+b^2},\dfrac{a^2b}{a^2+b^2}\right)}$ is the point on the line $\dfrac{x}{a}+\dfrac{y}{b}=1$ that ıs closest to the origin.

p.212, pr.85