Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (a) $\boxed{\text{10 Points}}$ $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}} = ?$ (**Do not use L'Hôpital's Rule**)

Solution:

$$\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}\frac{2+\sqrt{x}}{2+\sqrt{x}} = \lim_{x\to 4} \frac{x(4-x)(2+\sqrt{x})}{2^2-(\sqrt{x})^2} = \lim_{x\to 4} \frac{x\cancel{(4-x)}(2+\sqrt{x})}{\cancel{4-x}}$$

$$= \lim_{x\to 4} \frac{x(2+\sqrt{x})}{1} = 4(2+\sqrt{4}) = (4)(4) = \boxed{16}$$

p.55, pr.36

(b) $\boxed{\text{10 Points}}$ $\lim_{\theta\to 0} \sin\theta\cot(2\theta) = ?$ (**Do not use L'Hôpital's Rule**)

Solution:

$$\lim_{\theta\to 0} \sin\theta\cot(2\theta) = \lim_{\theta\to 0} \sin\theta\frac{\cos 2\theta}{\sin(2\theta)} = \lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}(\theta)(\cos(2\theta))\frac{1}{\frac{\sin(2\theta)}{2\theta}(2\theta)}\right]$$

$$= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\frac{\cancel{\theta}}{2\cancel{\theta}}\right)\left(\frac{1}{\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}}\right) = (1)(\frac{1}{2})(\frac{1}{1}) = \boxed{\frac{1}{2}}$$

p.73, pr.38

2. (a) $\boxed{\text{10 Points}}$ Define $h(2)$ in a way that extends $h(t) = \frac{t^2+3t-10}{t-2}$ to be *continuous* at $t = 2$.

Solution: First note that

$$\lim_{t\to 2}\frac{t^2+3t-10}{t-2} = \lim_{t\to 2}\frac{\cancel{(t-2)}(t+5)}{\cancel{t-2}} = \lim_{t\to 2}(t+5) = 2+5$$
$$= \boxed{7}$$

Therefore, the extension we want is

$$\boxed{\tilde{h}(t) = \begin{cases} \frac{t^2+3t-10}{t-2} & t \neq 2 \\ 7 & t = 2 \end{cases}}$$

Notice that $\lim_{t\to 2}\tilde{h}(t) = \lim_{t\to 2} h(t) = 7$ and this equals $\tilde{h}(2)$. This shows that $\tilde{h}(t)$ is continuous at $t = 2$.

p.83, pr.38

(b) $\boxed{\text{10 Points}}$ Determine all *asymptotes* of $y = \dfrac{x^2-4}{x-1}$. Give reason.

Solution: We first look for vertical asymptote(s). Equivalently, we ask if the line $x = 1$ is a vertical asymptote. In this case, we do because

$$\lim_{x\to 1^+}\frac{x^2-4}{x-1} = -\infty \qquad \lim_{x\to 1^-}\frac{x^2-4}{x-1} = +\infty$$

On the other hand,

$$\lim_{x\to\infty}\frac{x^2-4}{x-1} = \lim_{x\to\infty}\frac{x-4/x}{1-1/x} = +\infty \qquad \lim_{x\to-\infty}\frac{x^2-4}{x-1} = \lim_{x\to-\infty}\frac{x-4/x}{1-1/x} = -\infty$$

and so the graph has no horizontal asymptote. However, we often have oblique asymptote if there is no horizontal asymptote. In this case, we do because By long division of polynomials, we have

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2+0x-4 \\ & -x^2 + x \\ \hline & x-4 \\ & -x+1 \\ \hline & -3 \end{array}$$

Therefore, because we have

$$\frac{x^2-4}{x-1} = (x+1) + \frac{-3}{x-1}.$$

and $\lim_{x\to\pm\infty}\dfrac{-3}{x-1} = 0$ so we see that the line $y = x+1$ is an oblique asymptote.

p.487, pr.12

3. Suppose $f(x) = x^2 - 5$.

(a) $\boxed{\text{10 Points}}$ For $L = 11$, $x_0 = 4$ ve $\varepsilon = 1$, give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds. Give reason.

Solution: For $|f(x) - L| < \varepsilon$ to be satisfied we must have

$$|(x^2-5)-11| < 1 \Rightarrow |x^2-16| < 1 \Rightarrow -1 < x^2-16 < 1 \Rightarrow 15 < x^2 < 17 \Rightarrow \sqrt{15} < x < \sqrt{17}$$

The inequality holds for all in the open interval $(\sqrt{15}, \sqrt{17})$, so it holds for all $x \neq 4$ in this interval as well. But what we want is then to find a $\delta > 0$ to place the centered interval $4 - \delta < x < 4 + \delta$ (centered at $x_0 = 4$) inside the interval $(\sqrt{15}, \sqrt{17})$. To this end, we have

$$|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow 15 < x^2 < 17 \Rightarrow \sqrt{15} < x < \sqrt{17}$$

Then $-\delta + 4 = \sqrt{15} \Rightarrow \delta 4 - \sqrt{15} \approx 0.1270$, or $4 + \delta = \sqrt{17} \Rightarrow \delta = \sqrt{17} - 4 \approx 0.1231$; thus $\boxed{\delta \approx 0.12}$. The distance from 4 to the nearer endpoint of $(\sqrt{15}, \sqrt{17})$ is $\sqrt{17} - 4 \approx 0.1231$. If we take take $\delta = 0.1231$ or any smaller positive number, then the inequality $0 < |x-4| < \delta$ will automatically place x between $\sqrt{15}$ and $\sqrt{17}$ to make $|(x^2-5)-11| < 1$:

$$0 < |x-4| < 0.12 \Rightarrow |(x^2-5)-11| < 1.$$

p.695, pr.29

(b) 10 Points For the same function, using only the *definition of derivative* , find $f'(-1)$.

Solution: From the definition, we have

$$\begin{aligned} f'(-1) &= \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} \\ &= \lim_{h\to 0} \frac{(-1+h)^2-5-((-1)^2-5)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{1}-2h+h^2-\cancel{5}-\cancel{1}+\cancel{5}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(-2+h)}{\cancel{h}} \\ &= \lim_{h\to 0}(-2+h) = (-2+0) \\ &= \boxed{-2}. \end{aligned}$$

p.385, pr.88

4. (a) 10 Points $y = \sqrt{1+\cos(t^2)} \quad \Rightarrow \frac{dy}{dt} = ?$

Solution: Let $u = 1+\cos(t^2)$. Then $y = \sqrt{u} = u^{1/2}$. The derivatives of these respectively are $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ and $\frac{du}{dt} = -\left(\sin(t^2)\right)(2t) = -2t\sin(t^2)$ Therefore, by chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = \frac{1}{2\sqrt{u}}\cdot\left(-2t\sin(t^2)\right) = -2t\sin(t^2)\frac{1}{2\sqrt{u}} = \boxed{\frac{-t\sin(t^2)}{\sqrt{1+\cos(t^2)}}}$$

p.695, pr.31

(b) 10 Points Using *implicit differentiation*, find dy/dx if

$$5x^{4/5}+10y^{6/5} = 15.$$

Solution:

$$\frac{d}{dx}\left(5x^{4/5}+10y^{6/5}\right) = \frac{d}{dx}(16) \Rightarrow \frac{d}{dx}\left(5x^{4/5}\right)+\frac{d}{dx}\left(10y^{6/5}\right) = \frac{d}{dx}(16) \Rightarrow 5\frac{4}{5}x^{-1/5}+(10)\frac{6}{5}y^{1/5}\frac{dy}{dx} = 0.$$

So we obtain

$$-4x^{-1/5} = 12y^{1/5}\frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{3}\frac{x^{-1/5}}{y^{1/5}}} = -\frac{1}{3(xy)^{1/5}}$$

p.452, pr.24

5. (a) 10 Points The curves $y = x^2+ax+b$ and $y = cx-x^2$ have a *common tangent* line at $(1,0)$. Find a, b, and c.

Solution: Write $f(x) := x^2+ax+b$ and $g(x) := cx-x^2$. Then $f'(x) = 2x+a$ and $g'(x) = c-2x$ The curves $y = f(x)$ and $y = g(x)$ have a common tangent line at $(1,0)$ iff:
- They intersect there: $f(1) = g(1)$
- Their tangent lines have equal slope there: $f'(1) = g'(1)$.

$g(x) = cx-x^2$ passes through $(1,0)$, so $g(1) = 0 \Rightarrow 0 = c(1)-1 \Rightarrow c = 1$. Therefore $\boxed{g(x) = x-x^2}$. For this curve, $g'(1) = 1-2(1) = -1$. Since $g(x) = x-x^2$ and $f(x) = x^2+ax+b$ have common tangents at $x = 1$, $f(x)$ must have slope $m = -1$ at $x = 1$. Thus $f'(x) = 2x+a \Rightarrow -1 = 2(1)+a \Rightarrow a = -3 \Rightarrow f(x) = x^2-3x+b$.
Since this last curve passes though $(1,0)$, we have $0 = 1-3+b$ and so $b = 2$. We now do conclude that $a = -3$, $b = 2$ and $c = 1$ so the curves are $y = x^2-3x+2$ and $y = x-x^2$. Conversely, it is then easy to check that these (unique) curves satisfy the required conditions.

p.695, pr.31

(b) 10 Points Find the values of a that makes the following function *differentiable* for all x-values.

$$f(x) = \begin{cases} ax & x < 0 \\ x^2 - 3x & x \geq 0 \end{cases}$$

Solution: Assuming that the given function $f(x)$ is differentiable for all x-values, we only need to worry about the differentiability at $x = 0$. We then must have right-hand and left-hand derivatives must exist and equal each other at $x = 0$. The right-hand derivative is

$$\begin{aligned} f'_+(0) &= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{(0+h)^2 - 3(0+h) - 0}{h} \\ &= \lim_{h\to 0^+} \frac{h^2 - 3h}{h} \\ &= \lim_{h\to 0^+} \frac{\cancel{h}(h-3)}{\cancel{h}} \\ &= \lim_{h\to 0^+} (h-3) = -3 \end{aligned}$$

The left-hand derivative is

$$\begin{aligned} f'_-(0) &= \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{a(0+h) - 0}{h} \\ &= \lim_{h\to 0^-} \frac{ah}{h} \\ &= \lim_{h\to 0^-} \frac{\cancel{h}(a)}{\cancel{h}} \\ &= \lim_{h\to 0^-} (a) = a \end{aligned}$$

Hence equality forces $\boxed{a = -3}$. Conversely, when $a = -3$, the uniquely determined function

$$\boxed{f(x) = \begin{cases} -3x & x < 0 \\ x^2 - 3x & x \geq 0 \end{cases}}$$

is *differentiable* for all x-values.

p.452, pr.24