Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Problem	Points	Score
1	18	
2	22	
3	22	
4	16	
5	22	
Total:	100	

Do not write in the table to the right.

1. Given the curve $f(x) = \frac{8}{x^2}$ and the point $x_0 = 2$.

(a) 10 Points Using the definition, find the slope of this curve at x_0.

Solution: The required slope is

$$m = f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{8}{(2+h)^2} - \frac{8}{2^2}}{h}$$
$$= \lim_{h\to 0} \frac{32 - 8(2+h)^2}{4h(2+h)^2} = \lim_{h\to 0} \frac{\cancel{32} - \cancel{32} - 32h - 8h^2}{4h(2+h)^2}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(-32-8h)}{4\cancel{h}(2+h)^2}$$
$$= \frac{-32-8(0)}{4(2+0)^2} = -\frac{32}{16} = \boxed{-2}$$

p.583, pr.32

(b) 8 Points Find the equation for tangent line to the graph there.

Solution: The equation is $y - 2 = -2(x-2) \Rightarrow \boxed{y = -2x+6}$.

p.452, pr.24

2. Find the following limits, if they exist (**You are not allowed to use the L'Hôpital's Rule**).

(a) $\boxed{\text{10 Points}}$ $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

Solution:

$$\begin{aligned}
\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} &= \lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1}\frac{x}{x} = \lim_{x\to 1} \frac{1-x}{x(x-1)} \\
&= \lim_{x\to 1} \frac{-\cancel{(x-1)}}{x\cancel{(x-1)}} \\
&= \lim_{x\to 1} \frac{-1}{x} = \boxed{-1}
\end{aligned}$$

Alternatively, from the second definition of derivative, we have

$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \left[\frac{d}{dx}\frac{1}{x}\right]_{x=1} = \left[-\frac{1}{x^2}\right]_{x=1} = \left[-\frac{1}{(1)^2}\right]_{x=1} = \boxed{-1}$$

p.487, pr.6

(b) $\boxed{\text{12 Points}}$ $\lim\limits_{x\to 0} \dfrac{x\csc(2x)}{\cos(5x)}$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{x\csc(2x)}{\cos(5x)} &= \lim_{x\to 0} \frac{\frac{x}{\sin(2x)}}{\cos(5x)} = \lim_{x\to 0} \frac{x}{\sin(2x)}\frac{1}{\cos(5x)} \\
&= \frac{1}{2}\lim_{x\to 0} \frac{2x}{\sin(2x)}\frac{1}{\cos(5x)} = \frac{1}{2}(1)(1) = \boxed{\frac{1}{2}}
\end{aligned}$$

p.413, pr.58

3. (a) $\boxed{\text{12 Points}}$ Suppose $f(x) = \sqrt{x+1}$, $x_0 = 0$, $L = 1$, and $\varepsilon = \dfrac{1}{10}$. Give a value for $\delta > 0$ such that for all x with $0 < |x - x_0| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

Solution: First, we want:

$$\begin{aligned}
|\sqrt{x+1}-1| < 0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 &\Rightarrow 0.9 < \sqrt{x+1} < 1.1 \\
&\Rightarrow 0.81 < x+1 < 1.21 \\
&\Rightarrow -0.19 < x < 0.21
\end{aligned}$$

Moreover

$$|x-0| < \delta \Rightarrow -\delta < x < \delta.$$

Then $-\delta = -0.19$ or $\delta = 0.21$; thus, we choose $\delta = \min\{0.19, 0.21\} = \boxed{0.19}$.

p.573, pr.18

(b) $\boxed{\text{10 Points}}$ If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim\limits_{x\to 0} g(x)$. Explain your answer.

Solution: Notice that both $\lim\limits_{x\to 0}(2-x^2) = 2$ and $\lim\limits_{x\to 0}(2\cos x) = 2\cos 0 = 2$ exist and are equal to each other. Hence by Sandwich Theorem $\lim\limits_{x\to 0} g(x) = \boxed{2}$

p.56, pr.64

4. (a) $\boxed{\text{10 Points}}$ For what values of a is $f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$ is continuous at every x?

Solution: Suppose $f(x)$ is continuous at every x, hence in particular at $x = 2$. Then

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (a^2x - 2a) = 2a^2 - 2a$$

and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (12) = 12$$

must be equal to each other. Therefore $2a^2 - 2a = 12 \Rightarrow a^2 - a - 6 = 0 \Rightarrow (a+2)(a-3) = 0$. The values for a are $\boxed{-2}$ and $\boxed{3}$.

p.695, pr.37

(b) $\boxed{\text{6 Points}}$ Find the following limits, if they exist (**You are not allowed to use the L'Hôpital's Rule**).

$$\lim_{x\to 0^+} \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) =$$

Solution:

$$\lim_{x\to 0^+} \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) = +\infty - 1 = \boxed{\infty}$$

p.452, pr.24

$$\lim_{x\to 1^-} \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) =$$

Solution:

$$\lim_{x\to 1^-} \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) = 1 - \infty = \boxed{-\infty}$$

p.452, pr.24

5. (a) $\boxed{\text{12 Points}}$ The curve $y = ax^2 + bx + c$ passes through the point $(1,2)$ and is tangent to the line $y = x$ at the origin. Find a, b, and c.

Solution: Since the curve passes through $(1,2)$, we have $y(1) = 2 \Rightarrow a + b + c = 2$. The curve is tangent to $y = x$ says two things. First, the curve passes through the origin and so $y(0) = 0 \Rightarrow (a)(0)^2 + (b)(0) + c = 0 \Rightarrow c = 0$; next slope of this curve at the origin and that of $y = x$ do agree. This implies $y' = 2ax + b$ and so $y'(0) = b = 1$. The line $y = x$ has slope 1. So the curve at the origin must have slope 1. Moreover, the curve passes through $(1,2)$ yields the equation $a + b + c = 2$. But we have just found $c = 0$ and $b = 1$. Hence $a = 1$. Therefore the curve $\boxed{y = x^2 + x}$ has the given properties. p.583, pr.32

(b) $\boxed{\text{10 Points}}$ Find dy/dx if $y = \dfrac{1}{6}\left(1 + \cos^2(7x)\right)^3$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{6}\left(1+\cos^2(7x)\right)^3\right] &= \frac{1}{6}(3)\left(1+\cos^2(7x)\right)^2 \frac{d}{dx}\left(1+\cos^2(7x)\right) \\
&= \frac{1}{2}\left(1+\cos^2(7x)\right)^2 \left(0 + 2\cos(7x)\frac{d}{dx}(\cos(7x))\right) \\
&= \frac{1}{2}\left(1+\cos^2(7x)\right)^2 2\cos(7x)(-7\sin(7x)) \\
&= -7\sin(7x)\cos(7x)\left(1+\cos^2(7x)\right)^2 \\
&= \boxed{-\frac{7}{2}\sin(14x)\left(1+\cos^2(7x)\right)^2}
\end{aligned}$$

p.452, pr.24