Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	32	
2	35	
3	33	
Total:	100	

1. (a) $\boxed{\text{11 Points}}$ Suppose $f(x) = x^2$, $L = 4$, $c = -2$, $\varepsilon = 0.5$. Give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \varepsilon$ in the definition of limit.

Solution: Method 1: We want

$$|f(x) - L| = |x^2 - 4| < \varepsilon = 0.5 \Rightarrow -0.5 < x^2 - 4 < 0.5 \Rightarrow 3.5 < x^2 < 4.5 \Rightarrow \sqrt{3.5} < |x| < \sqrt{4.5}$$
$$\Rightarrow -\sqrt{4.5} < x < -\sqrt{3.5} \text{ for } x \text{ near } -2.$$

We also want

$$0 < |x - (-2)| < \delta \Rightarrow -\delta < x + 2 < \delta \Rightarrow -\delta - 2 < x < \delta - 2.$$
Then $-\delta - 2 = -\sqrt{4.5} \Rightarrow \delta = \sqrt{4.5} - 2 \approx 0.213$,
or $\delta - 2 = -\sqrt{3.5} \Rightarrow \delta = 2 - \sqrt{3.5} \approx 0.1292$;
thus $\delta = \sqrt{4.5} - 2 \approx 0.12$

Method 2: $L = \lim_{x \to c} f(x) = \lim_{x \to -2} x^2 = 4$. Consider $|f(x) - L| = |x^2 - 4| = |x + 2||x - 2|$, and suppose $\delta \leq 1$. Then $0 < |x - c| = |x + 2| < \delta$ implies $-1 < x + 2 < 1$ or $-3 < x < -1$, so adding -2 we get that $-5 < x - 2 < -3$ and $|x + 2||x - 2| < 5\delta$. But $5\delta = 0.5 = \frac{5}{10} = \frac{1}{2}$ if $\delta = \frac{1}{10} = 0.1$ and any value of $\delta \leq 0.1$ will work.

p.80, pr.23

(b) $\boxed{\text{10 Points}}$ If it exists, find the limit $\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}$.

Solution: By factoring the bottom, we have

$$\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \to 1} \frac{\cancel{1 - \sqrt{x}}}{\cancel{(1 - \sqrt{x})}(1 + \sqrt{x})} = \lim_{x \to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{1 + \sqrt{1}} = \boxed{\frac{1}{2}}$$

p.115, pr.11

(c) $\boxed{\text{11 Points}}$ Find the point(s) where the function $f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$ is continuous. Explain your answer.

Solution: For continuity at $x = 2$, we compute

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x - 2)}(x^2 + 2x + 4)}{\cancel{(x - 2)}(x + 2)} = \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{2^2 + (2)(2) + 4}{2 + 2} = \frac{12}{4} = 3.$$

Since $\lim_{x \to 2} f(x) = 3 = f(2)$, f is continuous at $x = 2$. But it is discontinuous at $x = -2$, since $\lim_{x \to 2} f(x)$ does not exist. Consequently, f is continuous at every $x \neq -2$.

p.98, pr.30

2. (a) $\boxed{\text{12 Points}}$ Find the derivative of $y = 2\sqrt{x}\sin\sqrt{x}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x}\sin\sqrt{x}) = 2\sqrt{x}\left(\cos\sqrt{x}\right)\frac{1}{2\sqrt{x}} + \left(\sin\sqrt{x}\right)\frac{2}{2\sqrt{x}} = \boxed{\cos\sqrt{x} + \frac{\sin\sqrt{x}}{\sqrt{x}}}$$

p.191, pr.22

(b) $\boxed{\text{12 Points}}$ Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1,1)$ and $(1,-1)$.

Solution:

$$x^3(3y^2\frac{dy}{dx}) + y^3(3x^2) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow 3x^3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 3x^2y^3$$

$$\Rightarrow \frac{dy}{dx}\left(3x^3y^2 + 2y - 1\right) = 1 - 3x^2y^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - 3x^2y^3}{3x^3y^2 + 2y - 1} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2}{4} == \frac{1}{2}$$

but $\left.\frac{dy}{dx}\right|_{(1,-1)}$ is undefined.

Therefore the curve has slope $-\frac{1}{2}$ at $(1,1)$ and at $(1,-1)$ has a vertical tangent because the slope is undefined there.

p.192, pr.87

(c) $\boxed{\text{11 Points}}$ Find the value of $\frac{dy}{dt}$ at $t = 0$ if $y = 3\sin(2x)$ and $x = t^2 + \pi$.

Solution:

$$x = t^2 + \pi \Rightarrow \frac{dx}{dt} = 2t; y = 3\sin(2x) \Rightarrow \frac{dy}{dx} = 6\cos(2x) = 6\cos(2(t^2+\pi)) = 6\cos(2t^2);$$

$$\text{thus } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \left(6\cos(2t^2)\right)\cdot(2t)$$

$$\Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = [6\cos(0)]\cdot(0) = \boxed{0}$$

p.192, pr.57

3. (a) $\boxed{\text{13 Points}}$ Find the values of a and b such that $f(x) = \dfrac{ax+b}{x^2-1}$ has a local extreme value 1 at $x = 3$. Is this local extreme value a local maximum, or a local minimum? Give reasons for your answer.

Solution:

$$f(x) = \frac{ax+b}{x^2-1} \Rightarrow f'(x) = \frac{a(x^2-1)-2x(ax+b)}{(x^2-1)^2} = \frac{-(ax^2+bx+a)}{(x^2-1)^2}$$

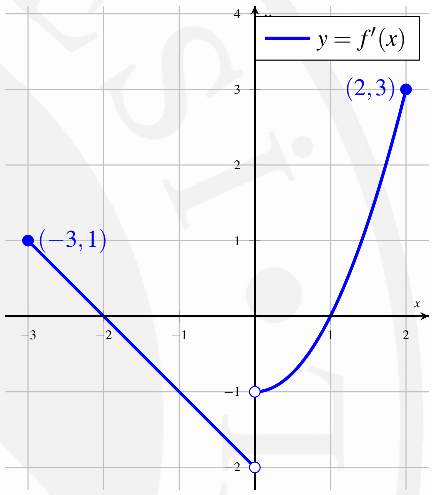
$$f'(3) = 0 \Rightarrow -\frac{1}{64}(9a+6b+a) = 0 \Rightarrow 5a+3b = 0.$$

We reqire also that $f(3) = 1$. Thus $1 = \dfrac{3a+b}{3^2-1} \Rightarrow 3a+b = 8$. Solving both equations yields $a = 6$ and $b = -10$. Now, $f'(x) = \dfrac{-2(3x-1)(x-3)}{(x^2-1)^2}$ so that

Thus f' changes sign from positive to negative so there is a local maximum at $x = 3$

p.257, pr.4

(b) $\boxed{\text{10 Points}}$ Figure on the right is the graph for a derivative $y = f'(x)$.



i. Write the intervals on which $y = f(x)$ is increasing.

Solution: The function f is increasing on $[-3,-2]$ and $[1,2]$, since over these intervals, the graph of f' is above x-axis.

p.258, pr.20

ii. Write the intervals on which $y = f(x)$ is decreasing.

Solution: The function f is decreasing on $[-2,0)$ and $(0,1]$, since over these intervals, the graph of f' is below x-axis.

p.258, pr.20

iii. Write the local maximum/minimum values of f and write the points where they occur.

Solution: The local maximum values occur only at $x = -2$ and $x = 2$; local minimum values occur only at $x = -3$ and $x = 1$ provided f is continuous at $x = 0$.

p.258, pr.20

(c) $\boxed{\text{10 Points}}$ Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0,1]$.

Solution: Let $f(x) = x^4 + 2x^2 - 2$. Then $f'(x) = 4x^3 + 4x$. Since $f(0) = -2 < 0$, $f(1) = 1 > 0$ and $f'(x) \geq 0$ for $0 \leq x \leq 1$, we may conclude from the Intermediate Value Theorem that $f(x) = 0$ has exactly one solution when $0 \leq x \leq 1$.

p.257, pr.13(a)