Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

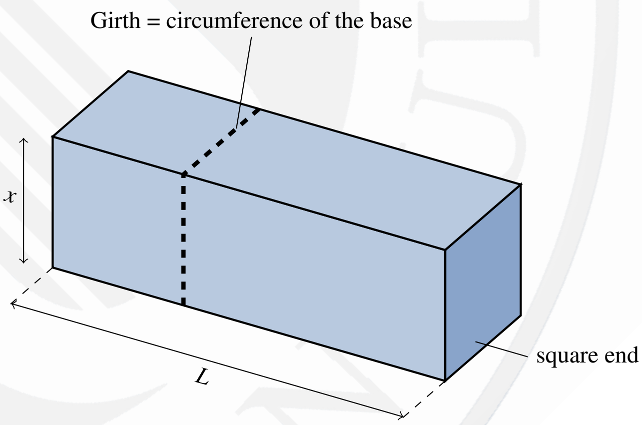
Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	22	
2	25	
3	27	
4	26	
Total:	100	

1. (a) $\boxed{\text{12 Points}}$ A postal service will accept packages only if the length plus girth is no more than 276 cm. (See the figure.) Assuming that the front face of the package (as shown in the figure) is square, what is the largest volume package that the postal service will accept?



Solution: We have $4x + L = 276$ and $V = x^2 L$.

The volume of the box is $V(x) = x^2(276 - 4x)$ where $0 \le x \le 69$. Then

$$V'(x) = 552x - 12x^2 = 12x(46 - x) = 0.$$

Critical points are 0 and 46, but $x = 0$ results in box.

Since $V''(x) = 552 - 24x < 0$ at $x = 46$, by second derivative test, we have a maximum.

The dimensions are $46 \times 46 \times 92$.

p.652, pr.3

(b) $\boxed{\text{10 Points}}$ Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_{\cos x}^{2} \frac{1}{\sqrt{1+t^3}}\, dt$.

Solution: First, notice that $y = \displaystyle\int_{\cos x}^{2} \frac{1}{\sqrt{1+t^3}}\, dt = -\int_{2}^{\cos x} \frac{1}{\sqrt{1+t^3}}\, dt$. Then

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\int_{2}^{\cos x} \frac{1}{\sqrt{1+t^3}}\, dt\right) = -\frac{d}{dx}\left(\int_{2}^{\cos x} \frac{1}{\sqrt{1+t^3}}\, dt\right)$$

Now let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. By the Chain Rule, we have

$$\frac{dy}{dx} = -\frac{d}{dx}\int_{2}^{u} \frac{1}{\sqrt{1+t^3}}\, dt = -\frac{d}{du}\left(\int_{2}^{u} \frac{1}{\sqrt{1+t^3}}\, dt\right)\frac{du}{dx} = -\frac{1}{\sqrt{1+u^3}}(-\sin x) = -\frac{1}{\sqrt{1+(\cos x)^3}}(-\sin x) = \boxed{\frac{\sin x}{\sqrt{1+\cos^3 x}}}$$

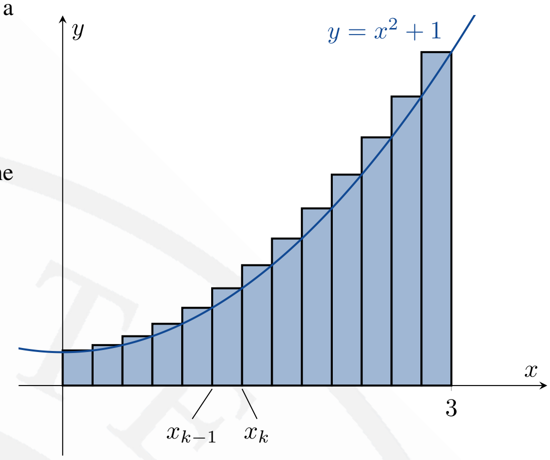
p.573, pr.18

2. (a) [15 Points] Find the area under $f(x) = x^2 + 1$ over $[0,3]$ as follows. First, find a formula, in terms of n, for the Riemann sum

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x$$

by dividing $[0,3]$ into n equal subintervals, and using $c_k = x_k$, the right-hand endpoint of $[x_{k-1}, x_k]$ for each k. Then find $\lim_{n\to\infty} S_n$.

Hint: You'll need $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$.



Solution: Let $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{n} = \dfrac{3}{n}$ and $c_k = k\Delta x = \dfrac{3k}{n}$. The right-hand sum is

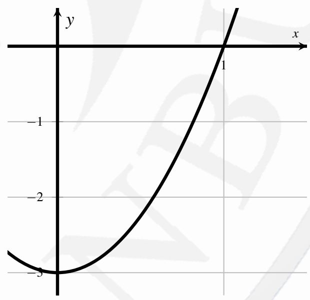
$$\begin{aligned}
\sum_{k=1}^{n} f(c_k)\,\Delta x &= \sum_{k=1}^{n}\left(c_k^2+1\right)\frac{3}{n} = \sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^2+1\right)\frac{3}{n} = \frac{3}{n}\sum_{k=1}^{n}\left(\frac{9k^2}{n^2}+1\right)\\
&= \frac{27}{n^3}\sum_{k=1}^{n}k^2 + \frac{3}{n}\sum_{k=1}^{n}(1)\\
&= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \left(\frac{3}{n}\right)(n) = \frac{27}{6}\frac{n}{n}\frac{n+1}{n}\frac{2n+1}{n} + 3 = \frac{9}{2}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 3.
\end{aligned}$$

Hence the area is

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+3\right) = \frac{9}{2}(1+0)(2+0)+3 = \frac{9}{2}(1)(2)+3 = \boxed{12}$$

p.83, pr.40

(b) [10 Points] Find the average value of $f(x) = 3x^2 - 3$ over the interval $[0,1]$.

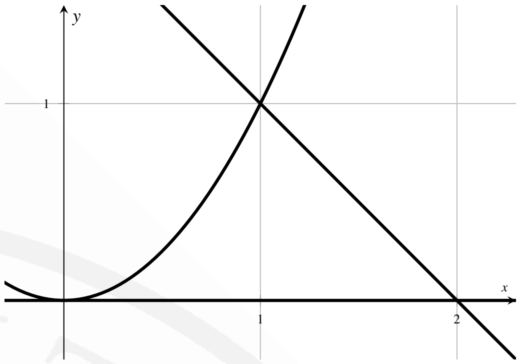


Solution: We have

$$\begin{aligned}
\mathrm{av}(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-0}\int_0^1 (3x^2-3)\,dx\\
&= 3\int_0^1 x^2\,dx - 3\int_0^1 dx\\
&= 3\left[\frac{x^3}{3}\right]_0^1 - 3\left[\frac{x}{1}\right]_0^1\\
&= 3\left(\frac{1^3}{3}\right) - 3(1-0) = \boxed{-2}
\end{aligned}$$

p.695, pr.37

3. (a) 13 Points Find the area of the region bounded by the parabola $y = x^2$, the line $x + y = 2$, and the x-axis.



Solution: We want the area between the x-axis and the curve $y = x^2$, $0 \le x \le 1$ *plus* the area of a triangle (formed by $x+y=2$, $x=1$, and $y=0$) with base 1 and height 1. Thus,

$$\begin{aligned}\text{TOTAL AREA} &= \int_0^1 x^2\,dx + \frac{1}{2}(1)(1)\\ &= \left[\frac{1}{3}x^3\right]_0^1 + \frac{1}{2}\\ &= \frac{1}{3} + \frac{1}{2} = \boxed{\frac{5}{6}}\end{aligned}$$

p.879, pr.42

(b) 14 Points Evaluate the integral $\int (x+5)(x-5)^{1/3}\,dx$.

Solution: Let $u = x-5$. Then $du = dx$ and $x+5 = u+10$. Therefore

$$\begin{aligned}\int (x+5)(x-5)^{1/3}\,dx = \int (u+10)(u)^{1/3}\,du = \int (uu^{1/3} + 10u^{1/3})\,du = \int (u^{4/3} + 10u^{1/3})\,du\\ = \left[\frac{u^{4/3+1}}{4/3+1} + 10\frac{u^{1/3+1}}{1/3+1}\right] + C = \frac{3}{7}u^{7/3} + \frac{15}{2}u^{4/3} + C\\ = \boxed{\frac{3}{7}(x-5)^{7/3} + \frac{15}{2}(x-5)^{4/3} + C}\end{aligned}$$

p.112, pr.26

4. Consider the function $y = \dfrac{x^3+x-2}{x-x^2} = -x-1+\dfrac{2x-2}{x-x^2}$. You may assume that $y' = \dfrac{2}{x^2} - 1$ and $y'' = -\dfrac{4}{x^3}$. Use this information to graph the function.

(a) 2 Points Identify the *domain* of f.

Solution: The domain of f is $\boxed{(-\infty,0)\cup(0,1)\cup(1,+\infty) = \mathbb{R} - \{0,1\}}$.

p.241, pr.45

(b) 7 Points Give the *asymptotes*.

Solution: For vertical asymptotes, there are two candidates: $x=0, x=1$. We have $\lim\limits_{x\to 1^+} \dfrac{x^3+x-2}{x-x^2} = \lim\limits_{x\to 1^\pm} \dfrac{\cancel{(x-1)}(x^2+x+2)}{\cancel{(x-1)}(-x)} =$ $\lim\limits_{x\to 1^\pm} \dfrac{x^2+x+2}{-x} = \dfrac{1^2+1+2}{-1} = -4 \ne \pm\infty$,
$\lim\limits_{x\to 0^-} \dfrac{x^3+x-2}{x-x^2} = +\infty$, $\lim\limits_{x\to 0^+} \dfrac{x^3+x-2}{x-x^2} = -\infty$. From these we see that the graph has *one vertical asymptote at* $x=0$. Next *there is no horizontal asymptote* as $\lim\limits_{x\to\pm\infty} \dfrac{x^3+x-2}{x-x^2} = \mp\infty$. But as

$$\frac{x^3+x-2}{x-x^2} = -x-1+\frac{2x-2}{x-x^2}$$

the line $y = -x-1$ is an oblique asymptote.

p.212, pr.85

(c) 5 Points Find the *intervals* where the graph is *increasing* and *decreasing*. Find the *local maximum* and *minimum* values.

Solution: We have $y' = -1 + \dfrac{2}{x^2} = 0$ if and only if $x^2 = 2$, that is iff $x = -\sqrt{2}$ and $x = \sqrt{2}$ are the critical points. Note that

$$y' \text{ is } \begin{cases} > 0, & \text{on } (-\sqrt{2}, 0) \cup (0, \sqrt{2}) \quad \text{y is increasing} \\ < 0, & \text{on } (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) \quad \text{y is decreasing} \end{cases}$$

Thus, y is decreasing on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ and increasing on $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$. Moreover, at the point $x = \sqrt{2}$ graph has a local local maximum and at $x = -\sqrt{2}$ graph has a local minimum. p.212, pr.85

(d) 5 Points Determine where the graph is *concave up* and *concave down*, and find any *inflection points*.

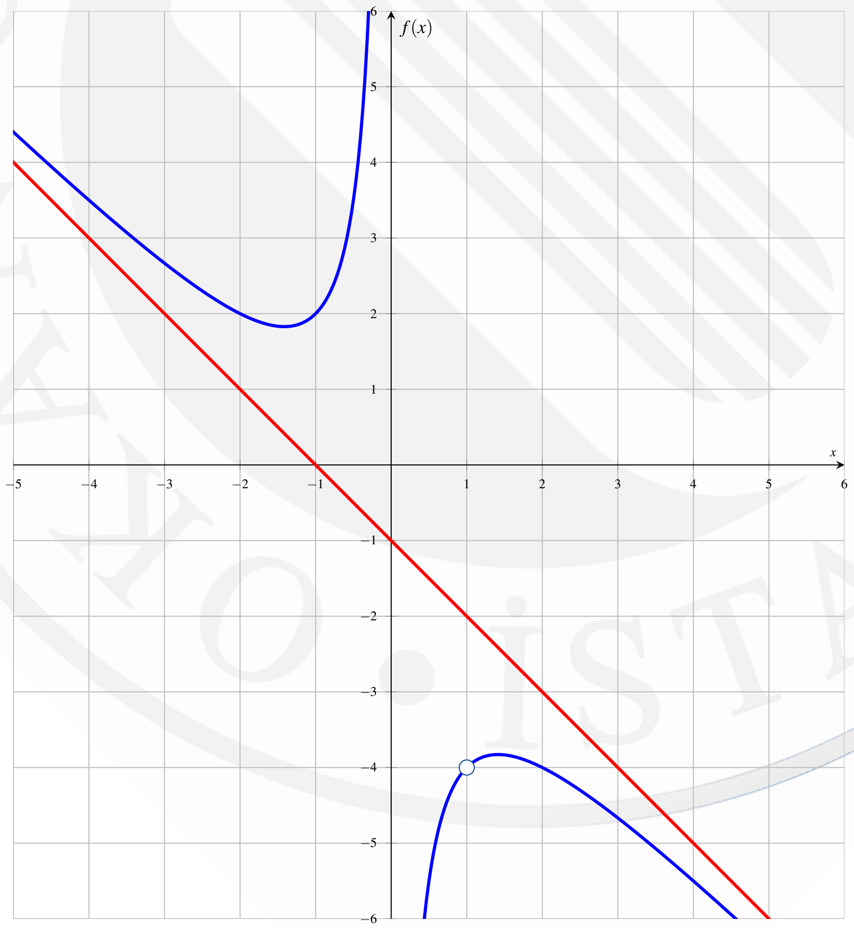
Solution: We have $y'' = -\dfrac{4}{x^3}$ and so

$$y'' \begin{cases} > 0, & \text{on } (-\infty, 0) \quad \text{y is concave up} \\ < 0, & \text{on } (0, \infty) \quad \text{y is concave down} \end{cases}$$

Hence f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. Also graph has *no point of inflection* there is no tangent line at $x = 0$. p.212, pr.85

(e) 7 Points *Sketch the graph* of the function. Label the asymptotes, critical points and the inflection points.

Solution:



p.212, pr.85