Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	33	
2	31	
3	36	
Total:	100	

1. (a) $\boxed{\text{10 Points}}$ Find the integral $\displaystyle\int_0^\pi 5(5-4\cos t)^{1/4}\sin t\,\mathrm{d}t$.

Solution: Let $u = 5-4\cos t$ and so $\mathrm{d}u = 4\sin t\,\mathrm{d}t$. When $t=0$, we have $u = 5-4\cos 0 = 1$ and when $t=\pi$, we have $u = 5-4\cos\pi = 9$. Hence

$$\begin{aligned}\int_0^\pi 5(5-4\cos t)^{1/4}\sin t\,\mathrm{d}t &= \frac{5}{4}\int_0^\pi (5-4\cos t)^{1/4}4\sin t\,\mathrm{d}t\\ &= \frac{5}{4}\int_1^9 u^{1/4}\,\mathrm{d}u = \frac{5}{4}\left[\frac{u^{5/4}}{5/4}\right]_1^9 = 9^{5/4}-1^{5/4} = \boxed{3^{5/2}-1}\end{aligned}$$

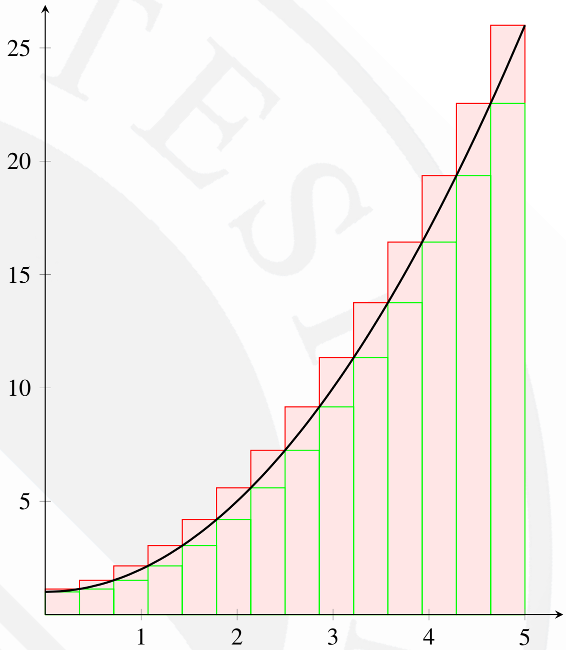
p.191, pr.22

(b) $\boxed{\text{13 Points}}$ Find the area under $f(x) = x^2+1$ over $[0,5]$ as follows. First, find a formula, in terms of n, for the Riemann sum

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x$$

by dividing $[0,5]$ into n equal subintervals, and using $c_k = x_k$, the right-hand endpoint of $[x_{k-1}, x_k]$ for each k. Then find $\lim_{n\to\infty} S_n$.

Hint: You'll need $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.



Solution: Let $\Delta x = \dfrac{b-a}{n} = \dfrac{5-0}{n} = \dfrac{5}{n}$ and $c_k = k\Delta x = \dfrac{5k}{n}$. The right-hand sum is

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\,\Delta x &= \sum_{k=1}^{n}\left(c_k^2+1\right)\frac{5}{n} = \sum_{k=1}^{n}\left(\left(\frac{5k}{n}\right)^2+1\right)\frac{5}{n} = \frac{5}{n}\sum_{k=1}^{n}\left(\frac{25k^2}{n^2}+1\right)\\ &= \frac{125}{n^3}\sum_{k=1}^{n}k^2 + \frac{5}{n}\sum_{k=1}^{n}(1)\\ &= \frac{125}{n^3}\frac{n(n+1)(2n+1)}{6} + \left(\frac{5}{n}\right)(n) = \frac{125}{6}\frac{n}{n}\frac{n+1}{n}\frac{2n+1}{n}+5 = \frac{125}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+5.\end{aligned}$$

Hence the area is

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{125}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+5\right) = \frac{125}{6}(1+0)(2+0)+5 = \frac{125}{6}(1)(2)+5 = \boxed{\frac{140}{3}}$$

p.192, pr.87

(c) $\boxed{\text{10 Points}}$ Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_{\sqrt{x}}^{0}\sin(t^2)\,\mathrm{d}t$.

Solution: First, notice that $y = \displaystyle\int_{\sqrt{x}}^{0}\sin(t^2)\,\mathrm{d}t = -\int_0^{\sqrt{x}}\sin(t^2)\,\mathrm{d}t$. Then

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\int_0^{\sqrt{x}}\sin(t^2)\,\mathrm{d}t\right) = -\frac{d}{dx}\left(\int_0^{\sqrt{x}}\sin(t^2)\,\mathrm{d}t\right)$$

Now let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. By the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= -\frac{d}{dx}\int_0^u \sin(t^2)\,\mathrm{d}t = -\frac{d}{du}\left(\int_0^u \sin(t^2)\,\mathrm{d}t\right)\frac{du}{dx} = -\sin(u^2)\left(\frac{1}{2\sqrt{x}}\right)\\ &= \boxed{-\frac{\sin x}{2\sqrt{x}}}\end{aligned}$$

p.192, pr.57

2. Suppose $y = \dfrac{x+1}{x-3} = 1 + \dfrac{4}{x-3}$. You may assume that $y' = -\dfrac{4}{(x-3)^2}$ and $y'' = \dfrac{8}{(x-3)^3}$.

(a) $\boxed{\text{11 Points}}$ Give the *asymptotes*.

Solution: First $\lim\limits_{x\to 3^+} \dfrac{x+1}{x-3} = +\infty$, $\lim\limits_{x\to 3^-} \dfrac{x+1}{x-3} = -\infty$. So the line $x = 3$ *is a vertical asymptote*. Since $\lim\limits_{x\to\pm\infty} \dfrac{x+1}{x-3} = 1$, the line $y = 1$ is a horizontal asymptote.

p.257, pr.4

(b) $\boxed{\text{10 Points}}$ Find the *intervals* where the graph is *increasing* and *decreasing*. Find the *local maximum* and *minimum* values. Determine the concavity and the points of inflection.

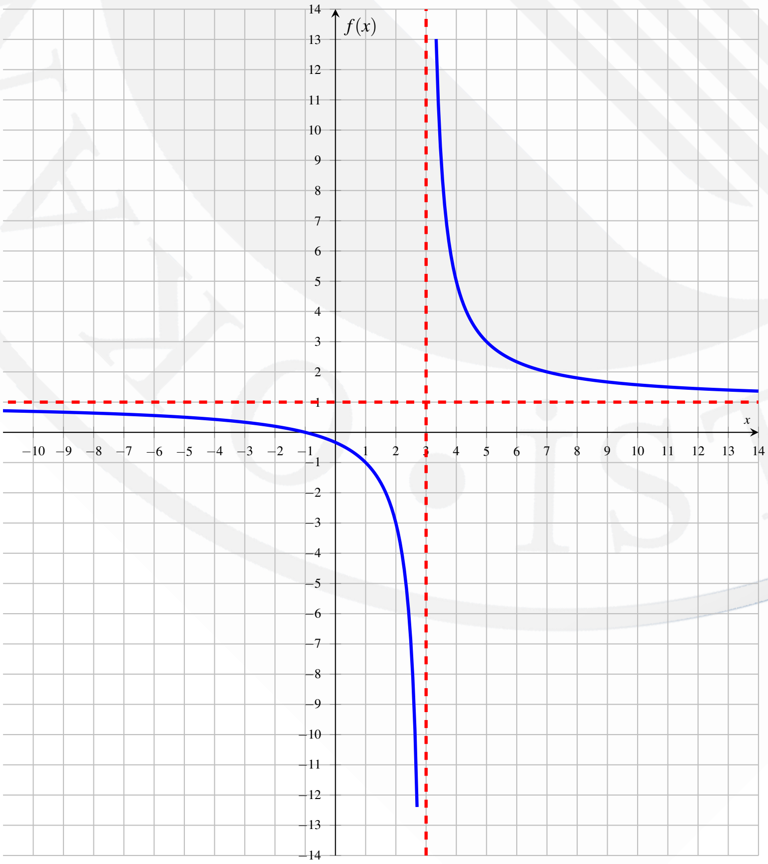
Solution: Since $y' = -\dfrac{4}{(x-3)^2} < 0$ for all $x \neq 3$, the graph is decreasing on $(-\infty, 3) \cup (3, \infty)$. Therefore no local maxima or minima occurs.

Moreover, we have $y'' = \dfrac{8}{(x-3)^3}$ and so

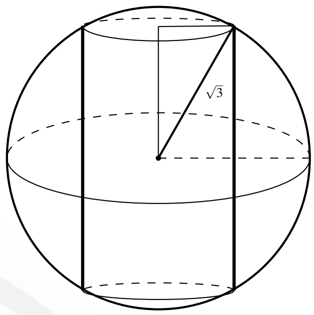
$$y'' \begin{cases} > 0, & \text{on } (3,\infty) \qquad \text{y is concave up} \\ < 0, & \text{on } (-\infty,3) \qquad \text{y is concave down} \end{cases}$$

Hence f is concave up on $(3,\infty)$ and concave down on $(-\infty,3)$. Also graph has *no point of inflection* as there is no tangent line at $x = 3$.

(c) $\boxed{\text{10 Points}}$ *Sketch the graph* of the function. Label the asymptotes, critical points and the inflection points.



3. (a) 10 Points Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius of $\sqrt{3}$.



Solution: Let r and h denote the radius and height of the cylinder respectively. Then, using trigonometry (right triangles - draw a line from the center to where the line touches the circle), you can find h in terms of r. From the diagram, we have $\left(\frac{h}{2}\right)^2 + r^2 = (\sqrt{3})^2 \Rightarrow r^2 = \frac{12-h^2}{4}$. The volume of the cylinder is

$$V = \pi r^2 h = \pi\left(\frac{12-h^2}{4}\right)h = \frac{\pi}{4}\left(12h - h^3\right), \quad \text{where } 0 \le h \le 2\sqrt{3}.$$

Then $V'(h) = \frac{3\pi}{4}(2+h)(2-h) \Rightarrow$ the critical points are -2 and 2, but $-2 \notin [0, 2\sqrt{3}]$. At $h=2$, there is a maximum since $V''(h) = -\frac{3\pi}{2}h$ is $V''(2) = -3\pi < 0$ a negative value. The dimensions of the largest cylinder are radius $= \boxed{\sqrt{2}}$ and height $= \boxed{2}$.

p.80, pr.23

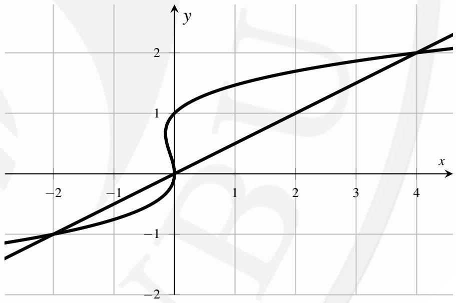
(b) 12 Points Find the integral $\int x^4(1-x^5)^{-1/5}\,\mathrm{d}x$.

Solution: Let $u = 1-x^5$ and so $du = -5x^4$. Then

$$\begin{aligned}\int x^4(1-x^5)^{-1/5}\,\mathrm{d}x &= -\frac{1}{5}\int (1-x^5)^{-1/5}(-5)x^4\,\mathrm{d}x \\ &= -\frac{1}{5}\int u^{-1/5}\,\mathrm{d}u = -\frac{1}{5}\frac{u^{-1/5+1}}{-1/5+1} + C = -\frac{1}{5}\frac{5}{4}u^{4/5} + C \\ &= \boxed{-\frac{1}{4}(1-x^5)^{4/5} + C}\end{aligned}$$

p.115, pr.11

(c) 14 Points From the figure, find the total area enclosed by $x = y^3 - y^2$ and $x = 2y$.



Solution: For $y \in [-1, 0]$, we have $y^3 - y^2 \ge 2y$ and for $y \in [0, 2]$, we have the reverse inequality $2y \ge y^3 - y^2$. Therefore

$$\begin{aligned}\text{TOTAL AREA} &= \int_{-1}^{0}(y^3 - y^2 - 2y)\ \mathrm{d}y + \int_0^2 (2y - y^3 + y^2)\ \mathrm{d}y \\ &= \left[\frac{1}{4}y^4 - \frac{1}{3}y^3 - y^2\right]_{-1}^{0} + \left[y^2 - \frac{1}{4}y^4 + \frac{1}{3}y^3\right]_0^2 \\ &= 0 - \frac{1}{4} - \frac{1}{3} + 1 + \left(4 - 4 + \frac{8}{3}\right) \\ &- 0 \\ &= \frac{5}{2} + \frac{8}{3} = \boxed{\frac{37}{12}}\end{aligned}$$

p.94, pr.10