Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor' s Name / Öğretim Üyesi

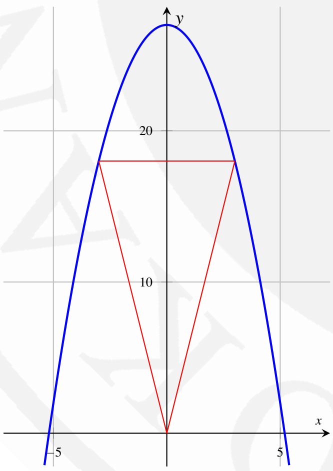
Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 75 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	30	
3	30	
4	15	
Total:	100	

1. (a) 15 Points An isoscales triangle has its vertex at the origin and its base paralel to the x-axis with the vertices above the x axis on the curve $y = 27 - x^2$. Find the largest possible area of the triangle.



Solution:

$$A = \frac{2x(27-x^2)}{2} = 27x - x^3$$
$$\frac{dA}{dx} = 27 - 3x^2 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \mp 3$$
$$A(-3) = -3(27-(-3)^2) = -54$$
$$A(3) = 3(27-3^2) = 54 \Rightarrow A = 54$$

(b) 10 Points For what value or values of constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly **one** horizontal tangent?

Solution:

$$y' = 3x^2 + 2kx + 3$$
$$\Rightarrow x^2 + \frac{2}{3}kx + = 0$$

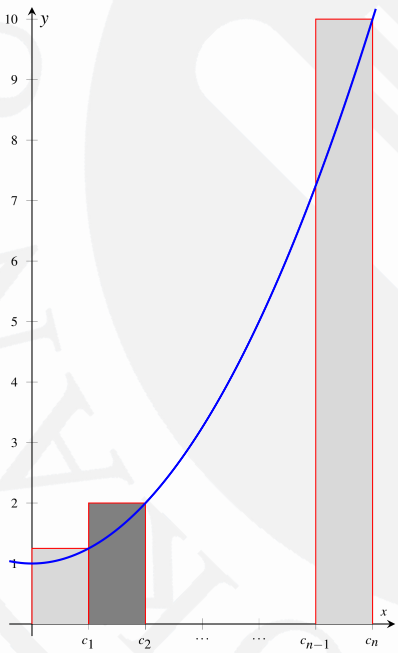
$$\triangle = b^2 - 4ac = 0$$
$$(\frac{2}{3}k)^2 - 4 = 0$$
$$k = \pm 3$$

2. (a) 15 Points Calculate $\dfrac{d}{dx}\displaystyle\int_2^{e^x} \frac{1}{\ln t}\,dt$.

Solution:
Let $u = e^x$. By the Chain Rule and the Fundamental Theorem of Calculus, it follows that

$$\frac{d}{dx}\int_2^{e^x}\frac{1}{\ln t}\,dt = \left(\frac{d}{du}\int_2^{u}\frac{1}{\ln t}\,dt\right)\left(\frac{du}{dx}\right)$$
$$= \left(\frac{1}{\ln u}\right)(e^x) = \frac{e^x}{\ln e^x} = \frac{e^x}{x}.$$

(b) 15 Points For the function $f(x) = x^2 + 1$, find a formula for the Riemann sum obtained by dividing the interval [0,3] into n equal subintervals and using the right hand point for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over [0,3].



Solution:

$$[a,b] = [0,3], \qquad \triangle x = \frac{b-a}{n} = \frac{3}{n}$$

$$c_k = x_k = a - \triangle x = 0 + k\frac{3}{n}$$

$$\sum_{k=1}^{n} f(x_k)\triangle x$$
$$= \sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^2 + 1\right)\frac{3}{n}$$
$$= \frac{27}{n^3}\sum_{k=1}^{n} k^2 + \frac{3}{n}\sum_{k=1}^{n} 1$$
$$= \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{3}{n}n$$
$$= \frac{9(2n^3+3n+n)}{n^3} + 3$$

$$A = \lim_{n\to\infty}\frac{9(2n^3+3n+n)}{n^3} + 3 = 12$$

3. (a) [10 Points] Find the average value of $y=\sqrt{3x}$ over $[0,3]$.

Solution:

$$av(f)=\frac{1}{b-a}\int_a^b f(x)dx$$

$$av(f)=\frac{1}{3-0}\int_0^3 \sqrt{3x}dx \qquad u=3x \Rightarrow du=3dx$$

$$=\frac{1}{9}\int_0^9 \sqrt{u}du \qquad u(0)=0,\ u(3)=9$$

$$=\left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^9=\frac{2}{27}\sqrt{9^3}=2$$

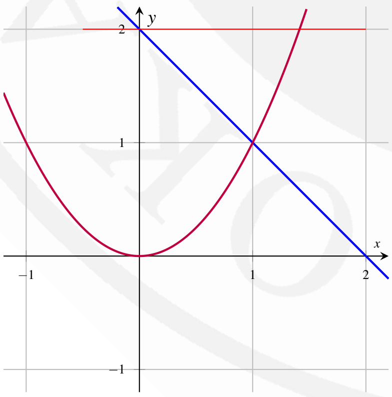
(b) [10 Points] Evaluate the integral $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$.

Solution:

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx=\int \frac{2du}{u^2} \qquad u=1+\sqrt{x},\ \frac{dx}{\sqrt{x}}=2du$$

$$=-2\frac{1}{u}+C=-2\frac{1}{1+\sqrt{x}}+C$$

(c) [10 Points] Find the area of the ”triangular”region bounded on the left by $x+y=2$, on the right by $y=x^2$, and above by $y=2$.



Solution:

$$y=2-x \qquad and \qquad y=x^2 \qquad intersect$$

$$y=x^2=2-x \Rightarrow (x-2)(x+1)=0$$

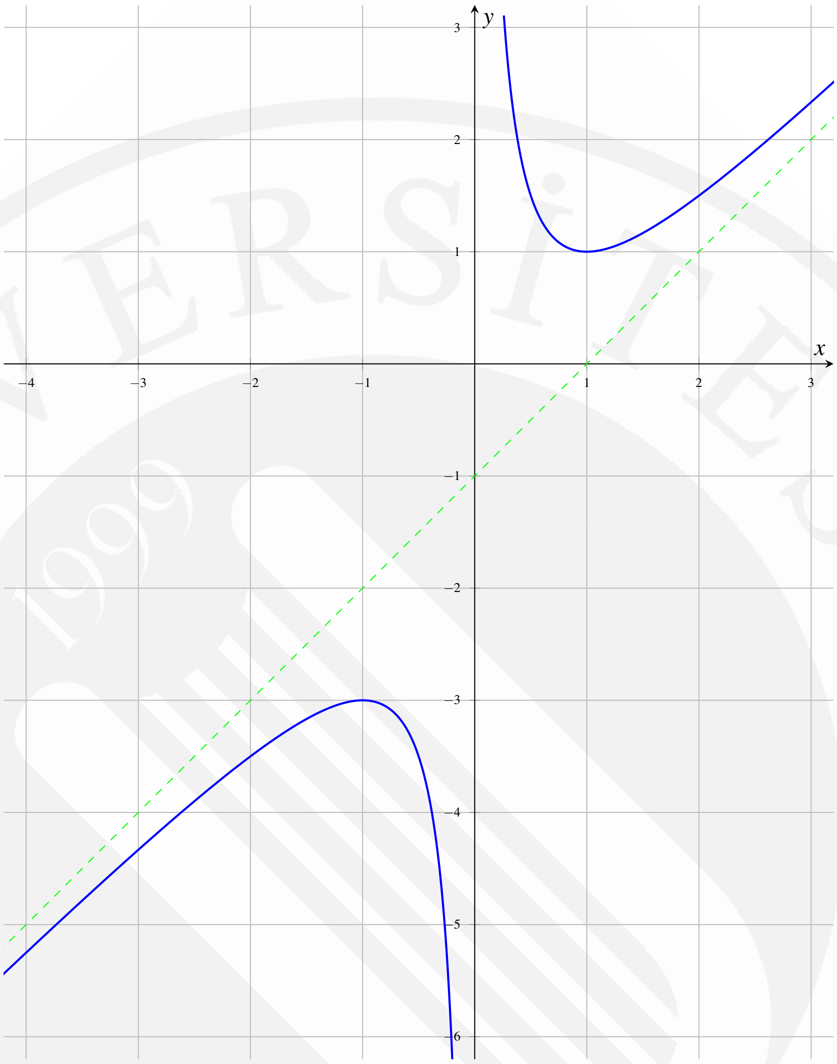
$$x=2 \Rightarrow y=1$$

$$\int_1^2 [\sqrt{y}-(2-y)]dy=\left.\frac{y^{\frac{3}{2}}}{\frac{3}{2}}-2y+\frac{y^2}{2}\right|_1^2$$

$$=\frac{4\sqrt{2}}{3}-4+2-(\frac{2}{3}-2+\frac{1}{2})=\frac{8\sqrt{2}+7}{6}$$

4. 15 Points Sketch the graph of the rational function $y = \dfrac{x^2 - x + 1}{x}$. Please note that $y' = 1 - \dfrac{1}{x^2}$ and $y'' = \dfrac{2}{x^3}$.

Solution:



Clearly y' is undefined at $x = 0$. We calculate that

$$0 = y' = 1 - \frac{1}{x^2} \implies x^2 = 1 \implies x = -1,\ 1.$$

Therefore the critical points are $x = -1$, $x = 0$ and $x = 1$. We can also see that y'' is not defined at $x = 0$ and $y'' \neq 0$ everywhere else. Hence the intervals to consider are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$.

Next we must find the asymptotes of the graph: Since $y = x - 1 + \frac{1}{x}$, we can see that $y \approx x - 1$ for large $|x|$. Hence $y = x - 1$ is an oblique asymptote of the graph. Moreover $x = 0$ is a vertical asymptote since $\lim_{x\to 0^-} x - 1 + \dfrac{1}{x} = -\infty$ and $\lim_{x\to 0^+} x - 1 + \dfrac{1}{x} = \infty$.

We can also say that

(i) f is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$;

(ii) f is decreasing on the intervals $(-1, 0)$ and $(0, 1)$;

(iii) f is concave down on the interval $(-\infty, 0)$;

(iv) f is concave up on the interval $(0, \infty)$; and

(v) $(-1, f(-1))$ is local maximum and $(1, f(1))$ is local minimum.

The table below summarises this information.

x	$-\infty$		-1		0		1		∞
$f'(x)$		+		-		-		+	
$f''(x)$		-		-		+		+	
$f(x)$