OKAN UNIVERSITY

Department of Mathematics

Math 216

Mathematics IV

Midterm Exam I

March 25, 2015

12:30-14:00

Surname : ____________________

Name : ____________________

ID # : ____________________

Department : ____________________

Section : ____________________

Instructor : ____________________

Signature : ____________________

- The exam consists of 6 questions.
- The duration of exam is 90 minutes.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.
- Do not leave the exam room during the first 20 minutes.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
10	20	15	15	20	20	100

1. (**10 points**) Classify the following differential equations.Write the order, linearity and the homogeneity of the equations.

a)

$$\frac{d^3y}{dx^3} + 2e^x \frac{d^2y}{dx^2} = x^3 + 5xy^2.$$

Solution: It is a third order, nonlinear, and non-homegeneous differential equation.

b)

$$x'' = (x')^2 + x' \sin t.$$

Solution: It is a second order, non-linear, and homegeneous differential equation.

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2. (**20 points**) Solve the following initial value problem.

$$y' - \tan x y = \sin 2x, \qquad y(0) = 2.$$

Solution: Note that the integrating factor is $\lambda(x) = e^{\int -\tan x dx} = e^{\ln|\cos x|} = \cos x$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$ Therefore, we get

$$\begin{aligned}(\cos x)\, y(x) &= \int \cos x \sin 2x dx = 2 \int \cos^2 x \sin x dx \Longrightarrow \\ y(x) &= -\frac{2\cos^2 x}{3} + \frac{C}{\cos x}\end{aligned}$$

Since $y(0) = 2$, it follows that

$$2 = y(0) = \frac{2\cos^2 0}{3} + \frac{C}{\cos 0} \Longrightarrow C = \frac{4}{3}.$$

Then, the solution of the initial value problem is

$$y(x) = \frac{2\cos^2 x}{3} + \frac{4}{3\cos x}.$$

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3. (**15 points**) Find the general solution of the following differential equation.

$$y' = (y-1)\cot x.$$

Solution: It is a separable differential equation. We solve the problem as follows.

$$\frac{dy}{dx} = (y-1)\cot x \Longrightarrow \frac{dy}{y-1} = \cot x dx$$
$$\Longrightarrow \int \frac{dy}{y-1} = \int \cot x dx + C = \int \frac{\cos x}{\sin x} dx + C$$
$$\Longrightarrow \ln(y-1) = \ln\sin x + C \Longrightarrow y - 1 = D\sin x \Longrightarrow y(x) = D\sin x + 1.$$

where $D = e^C$. Since this is a linear equation an alternative solution is given as follows.

$$\frac{dy}{dx} = (y-1)\cot x \Longrightarrow \frac{dy}{dx} - \cot xy = -\cot x.$$

Thus, the integrating factor is

$$\lambda = e^{\int -\cot x dx} = e^{-\ln(\sin x)} = \frac{1}{\sin x}.$$

Consequently, the general solution can be obtained as follows.

$$\frac{1}{\sin x} y(x) = \int -\frac{\cos x}{\sin^2 x} dx = \frac{1}{\sin x} + C.$$

This yields

$$y(x) = C\sin x + 1.$$

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4. (**15 points**) Write the linear, homogeneous, and constant coefficient differential equation whose general solution is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 \cos 2x + c_4 \sin 2x$$

Solution: The roots of the characteristic polynomial are 1, 1, 2i, -2i. The characteristic equation is

$$\begin{aligned} (r-1)^2(r^2+4) &= 0 \\ (r^2 - 2r + 1)(r^2+4) &= 0 \\ r^4 - 2r^3 + 5r^2 - 8r + 4 &= 0. \end{aligned}$$

Consequently, the differential equation is

$$y^{(4)} - 2y''' + 5y'' - 8y' + 4y = 0.$$

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5. (**20 points**) Solve the following initial value problem.

$$y'' - 2y' + y = 1 + 4e^t, y(0) = 3, y'(0) = 1.$$

Solution:
The characteristic equation is $r^2 - 2r + 1 = 0$ and its roots are $r_1 = r_2 = 1$. Therefore, the solution of the homogeneous equation is

$$y_H(t) = c_1 e^t + c_2 t e^t.$$

To find a particular solution, let us use the method of undetermined coefficients.

$$\begin{aligned} y_p &= A + Bt^2 e^t \\ y_p' &= 2Bte^t + Bt^2 e^t \\ y_p'' &= 2Be^t + 4Bte^t + Bt^2 e^t. \end{aligned}$$

Let us substitute these expressions in the differential equation. Then, we get

$$\begin{aligned} y'' - 2y' + y &= 1 + 4e^t \\ 2Be^t + 4Bte^t + Bt^2e^t - 4Bte^t - 2Bt^2e^t + A + Bt^2e^t &= 1 + 4e^t \\ A + (B - 2B + B)t^2e^t + (4B - 4B)te^t + 2Be^t &= 1 + 4e^t \end{aligned}$$

which implies that $A = 1, B = 2$ and $y_p = 1 + 2t^2e^t$. Thus, the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + 2t^2 e^t + 1.$$

Let us use the initial conditions to determine c_1 and c_2.

$$\begin{aligned} y(0) = 3 &\Rightarrow c_1 + 1 = 3 \Rightarrow c_1 = 2, \\ y'(0) = 1 &\Rightarrow c_1 + c_2 = 1 \Rightarrow c_2 = -1. \end{aligned}$$

Consequently, the solution of the initial value problem is

$$y(t) = 2e^t - te^t + 2t^2e^t + 1.$$

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6. (20 points) Solve the following initial value problem.

$$y'' - 2y' + y = \frac{e^t}{1+t^2}, y(0) = 3, y'(0) = 1.$$

Solution: The characteristic equation is $r^2 - 2r + 1 = 0$ and its roots are $r_1 = r_2 = 1$. Therefore, the solution of the homogeneous equation is

$$y_H(t) = c_1 e^t + c_2 t e^t.$$

To find a particular solution, let us use method of variation of the parameters. Assume that

$$y_p(t) = u_1 e^t + u_2 t e^t.$$

Then, we solve the following system of linear equations.

$$\begin{aligned} u_1' e^t + u_2' t e^t &= 0 \\ u_1' e^t + u_2' e^t + u_2' t e^t &= \frac{e^t}{1+t^2} \end{aligned}$$

If we substract the first equation from the second one, we obtain

$$u_2' e^t = \frac{e^t}{1+t^2} \Rightarrow u_2' = \frac{1}{1+t^2} \Rightarrow u_2 = \int \frac{1}{1+t^2} dt = \arctan t + c_{10}.$$

Since

$$\begin{aligned} u_1' e^t + u_2' t e^t &= 0 \\ u_1' + u_2' t &= 0 \\ u_1' &= -\frac{t}{1+t^2}, \end{aligned}$$

this implies that

$$u_1 = \int -\frac{t}{1+t^2} dt = -\frac{1}{2}\ln(1+t^2) + c_{20}.$$

Consequently, the general solution is

$$y(t) = c_{10} e^t + c_{20} t e^t - \frac{1}{2}\ln\left(t^2+1\right) e^t + t e^t \arctan t.$$

Since $y(0) = 3$, we get $c_1 = 3$. Since

$$y'(x) = c_1 e^t + c_2 (t+1) e^t - \frac{1}{2}\frac{2t}{1+t^2} e^t - \frac{1}{2}\ln\left(t^2+1\right) e^t + e^t \arctan t + t\frac{1}{1+t^2} e^t + t e^t \arctan t.$$

and $y'(0) = 1$, it follows that $c_1 + c_2 = 1$. Therefore, we get $c_2 = -2$ and $c_1 = 3$. Then, the solution of the initial value problem is

$$y(x) = 3e^t - 2te^t - \frac{1}{2} e^t \ln\left(1+t^2\right) + t e^t \arctan t.$$
