Your Name | Your Signature

Student ID #

Professor's Name | Your Department

- This exam is closed book.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	30	
3	20	
4	30	
Total:	100	

1. 20 points Find the solution of the following initial value problem by using Laplace Transformation.

$$y'' - 3y' - 4y = 0, \qquad y(0) = -1, y'(0) = 11$$

Solution: Let us calculate the Laplace transform of the differential equation above

$$\begin{aligned}
\mathscr{L}\{y'' - 3y' - 4y\} &= \mathscr{L}\{0\} \\
\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] - 3\left[s\mathscr{L}\{y\} - y(0)\right] - 4\mathscr{L}\{y\} &= 0 \\
\left[s^2\mathscr{L}\{y\} + s - 11\right] - 3\left[s\mathscr{L}\{y\} + 1\right] - 4\mathscr{L}\{y\} &= 0 \\
\left(s^2 - 3s - 4\right)\mathscr{L}\{y\} &= -s + 14 \\
\mathscr{L}\{y\} &= \frac{-s+14}{(s-4)(s+1)}
\end{aligned}$$

Let us calculate the inverse Laplace Transfonm to determine the $y(t)$ as follows.

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{\frac{-s+14}{(s-4)(s+1)}\right\} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{A}{s-4} + \frac{B}{s+1}\right\} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s-4} - \frac{3}{s+1}\right\} \\
y(t) &= 2e^{4t} - 3e^{-t}
\end{aligned}$$

2. $\boxed{\text{30 points}}$ Find the solution of the following initial value problem by using Laplace Transformation.

$$y'' + 2y' + 10y = e^{-t}, \qquad y(0) = 0, y'(0) = 0$$

Solution: Let us calculate the Laplace transform of the differential equation above

$$\begin{aligned}
\mathscr{L}\left\{y'' + 2y' + 10y\right\} &= \mathscr{L}\left\{e^{-t}\right\} \\
\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] + 2\left[s\mathscr{L}\{y\} - y(0)\right] + 10\mathscr{L}\{y\} &= \frac{1}{s+1} \\
s^2\mathscr{L}\{y\} + 2s\mathscr{L}\{y\} + 10\mathscr{L}\{y\} &= \frac{1}{s+1} \\
\left(s^2 + 2s + 11\right)\mathscr{L}\{y\} &= \frac{1}{s+1} \\
\mathscr{L}\{y\} &= \frac{1}{(s^2+2s+10)(s+1)}
\end{aligned}$$

Let us calculate the inverse Laplace Transfonm to determine the $y(t)$ as follows.

$$\begin{aligned}
y(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s^2+2s+10)(s+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{A}{s+1} + \frac{Bs+C}{s^2+2s+10}\right\} \\
y(t) &= \frac{1}{9}\mathscr{L}^{-1}\left\{\frac{1}{s+1} - \frac{s+1}{s^2+2s+10}\right\} \\
y(t) &= \frac{1}{9}\mathscr{L}^{-1}\left\{\frac{1}{s+1} - \frac{s+1}{(s+1)^2+9}\right\} \\
y(t) &= \frac{1}{9}\left(e^{-t} - e^{-t}\cos 3t\right)
\end{aligned}$$

3. $\boxed{\text{20 points}}$ Solve the following differential equation system.

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 2 & -1 \end{bmatrix} \mathbf{x}$$

Solution: Let us find the eigenvalues of the matrix above.

$$det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0$$

Thus, the eigenvalues are $\lambda_1 = 1+2i$ and $\lambda_2 = 1-2i$ and the corresponding eigenvectors are $v = [1+i \quad 1]^T$ and $\bar{v} = [1-i \quad 1]^T$, respectively. Let us calculate $e^{\lambda_1 t} v$

$$\begin{aligned} e^{\lambda_1 t} v &= e^{(1+2i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \\ &= e^t \left(\cos 2t + i \sin 2t\right) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \\ &= e^t \left(\begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{bmatrix} + i \begin{bmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{bmatrix} \right) \end{aligned}$$

Therefore, the general solution is

$$\mathbf{x} = e^t \left(c_1 \begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{bmatrix} \right)$$

4. $\boxed{\text{30 points}}$ Solve the following differential equation system.

$$\mathbf{x}' = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

Solution: Let us find the eigenvalues of the matrix above.

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ -3 & -5-\lambda \end{vmatrix} = (1-\lambda)(-5-\lambda) + 9 = \lambda^2 - 4\lambda + 4 = 0$$

Thus, the eigenvalues are $\lambda_1 = \lambda_2 = -2$ and the corresponding eigenvector is $v = [-1 \quad 1]^T$. And the generalized eigenvector is $w = [-1/3 \quad 0]^T$ Therefore, the general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} \right) e^{-2t}$$

Let us find the arbitrary constants.

$$\begin{aligned} -c_1 - \frac{c_2}{3} &= -6 \\ c_1 &= 4 \end{aligned}$$

Therefore $c_1 = 4$ and $c_2 = 6$. The particular solution is

$$\mathbf{x} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + 6 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} \right) e^{-2t}$$