Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any derivatives, integrals**.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	30	
3	25	
4	25	
Total:	100	

1. 20 Points Find the general solution of the differential equation $(3x^2+y^2)dx+2xydy=0$ which is homogeneous and exact.

Solution:

(a) **First Way**: The equation $2xydy+(3x^2+y^2)dx=0$ is an exact differential eq. Let us take $M(x,y)=3x^2+y^2$ and $N(x,y)=2xy$. The derivative of M with respect to y and the derivative of N with respect to x are $M_y=2y$ and $N_x=2y$, so the equation above is an exact differential equation. Therefore, there exists a function $F(x,y)=0$ such that $F_xdx+F_ydy=0$.

$$F_x=M=3x^2+y^2 \Rightarrow F(x,y)=\int(3x^2+y^2)dx=x^3+xy^2+h(y)$$

$$F_y=N=2xy \Rightarrow F_y=2xy+h'(y)=2xy \Rightarrow h'(y)=0 \Rightarrow h(y)=C$$

The solution is $F(x,y)=x^3+xy^2+C=0$

(b) **Second Way:** The equation above is a homogeneous differential equation, so we use the substitution $v=\frac{y}{x} \Rightarrow y=vx$ and $\frac{dy}{dx}=x\frac{dv}{dx}+v$.

$$\begin{aligned}
\frac{dy}{dx}=-\frac{3x^2+y^2}{2xy} \Rightarrow x\frac{dv}{dx}+v=-\frac{3x^2+v^2x^2}{2x^2v}&=-\frac{3+v^2}{2v}\\
\Rightarrow x\frac{dv}{dx}=-\frac{3+v^2}{2v}-v&=-\frac{3+3v^2}{2v}\\
\Rightarrow \frac{2v}{(1+v^2)}dv&=-3\frac{1}{x}dx\\
\Rightarrow \int\frac{2v}{(1+v^2)}dv&=-3\int\frac{1}{x}dx\\
\Rightarrow \ln|1+v^2|&=-3\ln|x|+\ln C\\
\Rightarrow 1+v^2&=Cx^{-3}\\
\Rightarrow 1+\frac{y^2}{x^2}&=\frac{C}{x^3}\\
\Rightarrow x^3+xy^2&=C
\end{aligned}$$

2. (a) $\boxed{\text{15 Points}}$ Find the solution of the initial value problem $y'' - 3y' - 10y = 0$ and $y(0) = 7, y'(0) = 7$.

Solution:

(a) **First Way:** The char. eq. of the differential equation above is $r^2 - 3r - 10 = 0$. The roots are $r_1 = 5$ and $r_2 = -2$. The general solution is $y(t) = c_1 e^{5t} + c_2 e^{-2t}$. Let us calculate c_1 and c_2.

$$\begin{array}{l} c_1 + c_2 = 7 \\ 5c_1 - 2c_2 = 7 \end{array} \Rightarrow \begin{array}{l} c_1 = 3 \\ c_2 = 4 \end{array}$$

Therefore the solution is $y(t) = 3e^{5t} + 4e^{-2t}$.

(b) **Second Way:** Let us solve same question by using Laplace Transformation.

$$\begin{aligned} \mathscr{L}\{y'' - 3y' - 10y\} &= \mathscr{L}\{0\} \\ (s^2\mathscr{L}\{y\} - sy(0) - y'(0)) - 3(s\mathscr{L}\{y\} - y(0)) - 10\mathscr{L}\{y\} &= 0 \\ s^2\mathscr{L}\{y\} - 7s - 7 - 3s\mathscr{L}\{y\} + 21 - 10\mathscr{L}\{y\} &= 0 \\ (s^2 - 3s - 10)\mathscr{L}\{y\} &= 7s + 7 - 21 \\ \mathscr{L}\{y\} = \frac{7s-14}{(s^2-3s-10)} = \frac{7s-14}{(s-5)(s+2)} &= \frac{3}{s-5} + \frac{4}{s+2} \\ y(t) = \mathscr{L}^{-1}\left\{\frac{3}{s-5} + \frac{4}{s+2}\right\} \\ y(t) = 3e^{5t} + 4e^{-2t} \end{aligned}$$

(c) $\boxed{\text{15 Points}}$ Find the solution of $y'' + 4y' + 5y = -2e^{-2t}$, $y(0) = 0$ and $y'(0) = 0$.

Solution:

(a) **First Way:** The characteristic equation of the d,ifferential eq. above is $r^2 + 4r + 5 = (r+2)^2 + 1 = 0$. Therefore, the roots of the char. eq. are $r_1 = -2 + i$, $r_2 = -2 - i$ and $y_h(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$. Let us find y_p by using method of undetermined coefficients.

$$y_p(t) = Ae^{-2t} \Rightarrow y_p'(t) = -2Ae^{-2t} \quad \text{and} \quad y_p''(t) = 4Ae^{-2t}$$

$$\begin{aligned} y_p'' + 4y_p' + 5y_p &= -2e^{-2t} \\ 4Ae^{-2t} + 4(-2Ae^{-2t}) + 5Ae^{-2t} &= -2e^{-2t} \\ Ae^{-2t} = -2e^{-2t} \Rightarrow A = -2 \Rightarrow y_p &= -2e^{-2t} \end{aligned}$$

The general solution of the problem is $y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) - 2e^{-2t}$. Let us use the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$\begin{aligned} y(0) = 0 \Rightarrow y(0) = e^0(c_1 \cos 0 + c_2 \sin 0) - 2e^0 = 0 \Rightarrow c_1 = 2 \\ y'(0) = 0 \Rightarrow y'(t) = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t) + 4e^{-2t} \\ \Rightarrow y'(0) = -2e^0(c_1 \cos 0 + c_2 \sin 0) + e^0(-c_1 \sin 0 + c_2 \cos 0) + 4e^0 = 0 \Rightarrow -2c_1 + c_2 + 4 = 0 \Rightarrow c_2 = 0 \end{aligned}$$

The solution of the initial value problem is $y(t) = 2e^{-2t}\cos t - 2e^{-2t}$.

(b) **Second Way :** Let us use Laplace Transformation to solve differential equation.

$$(\mathrm{s}^2 + 4s + 5)\mathscr{L}\{y\} = \frac{-2}{s+2}$$

3. $\boxed{\text{25 Points}}$ Find the solution of the following system of differential equations $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = \lambda_3 = 1$ and $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Solution: The eigenvalues of the system are given at the question as $\lambda_1 = 3$ and $\lambda_2 = \lambda_3 = 1$. Let us find the corresponding eigenvectors.

$$(A-3I)\mathbf{u} = \mathbf{0} \Rightarrow \begin{bmatrix} 2-3 & 1 & 1 \\ 0 & 1-3 & 0 \\ 1 & -1 & 2-3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$(A-I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 2-1 & 1 & 1 \\ 0 & 1-1 & 0 \\ 1 & -1 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

We obtain the third one by solving the system $(A-I)\mathbf{w} = \mathbf{v}$.

$$(A-I)\mathbf{w} = \mathbf{v} \Rightarrow \begin{bmatrix} 2-1 & 1 & 1 \\ 0 & 1-1 & 0 \\ 1 & -1 & 2-1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{t}$. Let us compute arbitrary coefficients c_1 and c_2

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \Rightarrow \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{0} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} 0 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{0} \Rightarrow \begin{matrix} c_1 + c_2 = 1 \\ c_3 = -2 \\ c_1 - c_2 = 3 \end{matrix} \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = -1 \\ c_3 = -2 \end{matrix}$$

$$\Rightarrow \mathbf{x}(t) = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} - 2 \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{t} \Rightarrow \mathbf{x}(t) = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} -2t-1 \\ -2 \\ 2t+1 \end{bmatrix} e^{t}$$

4. $\boxed{\text{25 Points}}$ Find the solution of the following system of differential equations.

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 7 \\ 4e^{-t} - 3 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution: Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) - 0 \Rightarrow \lambda_1 = -3, \lambda_2 = 2.$$

The eigenvector corresponding to $\lambda_1 = -3$ is obtained by solving the system $(A + 3I)\mathbf{v} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 2+3 & 1 & 0 \\ 0 & -3+3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 5 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

The eigenvector corresponding to $\lambda_2 = 2$ is obtained by solving the system $(A - 2I)\mathbf{w} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 2-2 & 1 & 0 \\ 0 & -3-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}_h(t) = c_1 \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}.$$

Let us find $\mathbf{x}_p$ by using method of undetermined coefficients.

$$\mathbf{x}_p = \begin{bmatrix} Ae^{-t} + B \\ Ce^{-t} + D \end{bmatrix} \Rightarrow \mathbf{x}_p' = \begin{bmatrix} -Ae^{-t} \\ -Ce^{-t} \end{bmatrix}$$

$$\Rightarrow \mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 6 \\ 4e^{-t} - 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -Ae^{-t} \\ -Ce^{-t} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} Ae^{-t} + B \\ Ce^{-t} + D \end{bmatrix} + \begin{bmatrix} 7 \\ 4e^{-t} - 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -Ae^{-t} \\ -Ce^{-t} \end{bmatrix} = \begin{bmatrix} 2Ae^{-t} + 2B + Ce^{-t} + D + 7 \\ -3Ce^{-t} - 3D + 4e^{-t} - 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -Ae^{-t} \\ -Ce^{-t} \end{bmatrix} = \begin{bmatrix} 2A + C \\ -3C + 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 2B + D + 7 \\ -3D - 3 \end{bmatrix}$$

$$-A = 2A + C + 1 \qquad -C = -3C + 4 \qquad 2B + D + 7 = 0 \qquad -3D - 3 = 0 \Rightarrow A = -1, B = -3C = 2, D = -1$$

$$\Rightarrow \mathbf{x}_p = \begin{bmatrix} -e^{-t} - 3 \\ 2e^{-t} - 1 \end{bmatrix}$$

The general solution of the system is $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -e^{-t} - 3 \\ 2e^{-t} - 1 \end{bmatrix}$ Let us use initial value $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -11 \end{bmatrix}$.

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^0 + \begin{bmatrix} -e^0 - 3 \\ 2e^0 - 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 - 3 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 - 4 \\ -5c_1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = \frac{1}{5} \qquad c_2 = \frac{19}{5}$$

The solution of thwe system is $\mathbf{x} = \frac{1}{5} \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-3t} + \frac{19}{5} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -e^{-t} - 3 \\ 2e^{-t} - 1 \end{bmatrix}$