Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any derivatives, integrals**.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 75 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	30	
3	25	
4	25	
Total:	100	

1. (a) 10 Points Show that $ydx + (4xy - e^{-4y})dy = 0$ is not an exact differential equation and find the integrating factor $\lambda(y)$.

Solution: Let us take $M(x,y) = y$ and $N(x,y) = 4xy - e^{-4y}$. The derivative of M with respect to y is $M_y = 1$ and $N_x = 4y$. $M_y \neq N_x$, so it is not an exact differential equation. Let us find the integrating factor as

$$\lambda(y) = e^{\int \frac{M_y - N_x}{-M} dy} = e^{\int \frac{4y-1}{y} dy} = e^{\int (4 - \frac{1}{y}) dy} = \frac{e^{4y}}{y}$$

(b) 10 Points Find the general solution of the $\lambda(y)\left[ydx + (4xy - e^{-4y})dy = 0\right]$.

Solution: The equation $e^{4y}dx + (4xe^{4y} - \frac{1}{y})dy = 0$ is an exact equation. therefore, there exists a function $F(x,y) = 0$ such that $F_x dx + F_y dy = 0$. $F_x = e^{4y}$ and $F_y = 4xe^{4y} - \frac{1}{y}$.

$$F(x,y) = \int e^{4y} dx = xe^{4y} + h(y)$$

$$F_y = 4xe^{4y} + h'(y) = 4xe^{4y} - \frac{1}{y} \Rightarrow h'(y) = -\frac{1}{y} \Rightarrow h(y) = -\ln|y| + C$$

$$F(x,y) = xe^{4y} - \ln|y| + C = 0$$

2. (a) $\boxed{\text{15 Points}}$ Solve the initial value problem $t^2y' + 2ty + \tan t = 0$, $y(2\pi) = -2$.

Solution: Let us write the differential equation as $y' + \frac{2}{t}y = -\frac{\tan t}{t^2}$. It is a linear differential equation, so we calculate the integrating factor as

$$\lambda(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

The general solution of the given differential obtained by solving

$$t^2y(t) = -\int t^2 \frac{\tan t}{t^2} dt = \ln|\cos t| + C \Rightarrow y(t) = \frac{\ln|\cos t|}{t^2} + \frac{C}{t^2}.$$

Let us use the initial value to find C.

$$y(2\pi) = \frac{\ln|\cos 2\pi|}{4\pi^2} + \frac{C}{4\pi^2} = -2 \Rightarrow C = -8\pi^2.$$

The particular solution of the differential equation is $y(t) = \frac{\ln|\cos t|}{t^2} - \frac{8\pi^2}{t^2}$

(b) $\boxed{\text{15 Points}}$ Solve the initial value problem $\frac{dy}{dx} = \frac{7x}{y + x^2y}$, $y(0) = 5$.

Solution:

$$\frac{dy}{dx} = \frac{7x}{y+x^2y} \Rightarrow \frac{dy}{dx} = \frac{7x}{y(1+x^2)} \Rightarrow ydy = \frac{x}{1+x^2}dx \Rightarrow \int ydy = \int \frac{7x}{1+x^2}dx \Rightarrow \frac{y^2}{2} = \frac{7}{2}\ln|1+x^2| + C$$

Let us calculate the arbitrary constant by using initial value.

$$y(0) = 5 \Rightarrow \frac{5^2}{2} = \frac{7}{2}\ln 1 + C \Rightarrow C = \frac{25}{2}$$

$$y^2 = 7\ln|1+x^2| + 25$$

3. $\boxed{\text{25 Points}}$ Find the general solution of $y''' + y' = 6\cos t - 4\sin t + e^{-2t}$

Solution: The characteristic equation of the $y''' + y' = 0$ is $r^3 + r = 0 \Rightarrow r(r^2+1) = 0$ and the roots are $r_1 = 0, r_2 = i, r_3 = -i$. The general solution of the homogeneous equation is

$$y_h(t) = c_1 e^{0t} + c_2\cos t + c_3\sin t \Rightarrow y_h(t) = c_1 + c_2\cos t + c_3\sin t$$

Let us calculate the particular solution y_p by using method of undetermined coefficients. Let us take $y_p = (A\cos t + B\sin t)t + Ce^{-2t}$. The derivatives of the y_p are as follows.

$$\begin{aligned}
y_p' &= (A\cos t + B\sin t) + (-A\sin t + B\cos t)t - 2Ce^{-2t} = (Bt + A)\cos t + (-At + B)\sin t - 2Ce^{-2t} \\
y_p'' &= 2(-A\sin t + B\cos t) + (-A\cos t - B\sin t)t + 4Ce^{-2t} = (-At + 2B)\cos t + (-Bt - 2A)\sin t + 4Ce^{-2t} \\
y_p''' &= 3(-A\cos t - B\sin t) + (A\sin t - B\cos t)t - 8Ce^{-2t} = (-Bt - 3A)\cos t + (At - 3B)\sin t - 8Ce^{-2t}
\end{aligned}$$

Let us substitute them.

$$\begin{aligned}
[(-Bt - 3A)\cos t + (At - 3B)\sin t - 8Ce^{-2t}] + [(Bt + A)\cos t + (-At + B)\sin t - 2Ce^{-2t}] &= 6\cos t - 4\sin t + e^{-2t} \\
-2A\cos t - 2B\sin t - 10Ce^{-2t} &= 6\cos t - 4\sin t + e^{-2t} \\
\Rightarrow A = -3, B = 2, C = -\frac{1}{10} \Rightarrow y_p = (-3\cos t + 2\sin t)t - \frac{1}{10}e^{-2t}
\end{aligned}$$

The general solution of the given differential equation is

$$y(t) = c_1 + c_2\cos t + c_3\sin t + (-3\cos t + 2\sin t)t - \frac{1}{10}e^{-2t}$$

4. $\boxed{\text{25 Points}}$ Find the general solution of $y'' - 2y' + y = e^x \ln x$, $x > 0$.

Solution: The char. eq. is $r^2 - 2r + 1 = 0$ and its roots are $r_1 = r_2 = 1$. The general solution of the homogeneous eq. is

$$y_h = c_1 e^x + c_2 x e^x$$

The particular solution is $y_p = u_1 e^x + u_2 x e^x$ and we solve the system

$$\begin{aligned} u_1' e^x + u_2' x e^x &= 0 \\ u_1' e^x + u_2'(e^x + x e^x) &= e^x \ln x \end{aligned}$$

Therefore $u_1' = -x \ln x$ and $u_2' = \ln x$ and we can calculate u_1 and u_2 by using integration by parts easily
$u_1 = -\int x \ln x dx = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2$ and $u_2 = \int \ln x dx = x \ln x - x$.
$y_p = \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right) e^x + (x \ln x - x) x e^x = \left(\frac{3}{2}\ln x - \frac{5}{4}\right) x^2 e^x$
The general solution is

$$y(x) = c_1 e^x + c_2 x e^x + \left(\frac{1}{2}\ln x - \frac{3}{4}\right) x^2 e^x$$