Your Name / Adınız - Soyadınız	Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi	Your Department / Bölüm

- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any derivatives, integrals**.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) 10 Points Find the Laplace Transformation of $f(t) = 4e^{2t} - 3\cos 3t + 7t^3$,

Solution:

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{4e^{2t} - 3\cos 3t + 7t^3\right\} = 4\frac{1}{s-2} - 3\frac{s}{s^2+9} + \frac{7.3!}{s^4} = \frac{4}{s-2} - \frac{3s}{s^2+9} + \frac{42}{s^4}, \quad s > 2$$

(b) 15 Points Find the Laplace Transformation of $f(t) = 4t^3e^{-2t} - 3\sin 3t$.

Solution: Remember that the $\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k)$ where $F(s) = \mathscr{L}\{f(t)\}$. Let us calculate $\mathscr{L}\left\{e^{-2t}t^3\right\}$ as follows

$$\mathscr{L}\left\{t^3\right\} = \frac{3!}{s^4} \Rightarrow \mathscr{L}\left\{e^{-2t}t^3\right\} = \frac{3!}{(s+2)^4}$$

and

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{4t^3e^{-2t} - 3\sin 3t\right\} = 4\mathscr{L}\left\{t^3e^{-2t}\right\} - 3\mathscr{L}\{\sin 3t\} = 4\frac{3!}{(s+2)^4} - 3\frac{3}{s^2+9} = \frac{24}{(s+2)^4} - \frac{9}{s^2+9}, \quad s > 0$$

2. (a) $\boxed{\text{10 Points}}$ Find the Inverse Laplace Transformation of $F(s) = \dfrac{4s^2 - 12}{(s-1)(s+1)(s+3)}, \quad s > 1$

Solution:

$$\frac{4s^2-12}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$
$$4s^2 - 12 = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1)$$
$$s = 1 \Rightarrow -8 = 8A \Rightarrow A = -1$$
$$s = -1 \Rightarrow -8 = -4B \Rightarrow B = 2$$
$$s = -3 \Rightarrow 24 = 8C \Rightarrow C = 3$$
$$\mathscr{L}^{-1}\left\{\frac{4s^2-12}{(s-1)(s+1)(s+3)}\right\} = \mathscr{L}^{-1}\left\{\frac{-1}{s-1} + \frac{2}{s+1} + \frac{3}{s+3}\right\} = -e^t + 2e^{-t} + 3e^{-3t}$$

(b) $\boxed{\text{15 Points}}$ Find the Inverse Laplace Transformation of $F(s) = \dfrac{s+1}{s^2+4s+8}$, $s > 0$.

Solution:

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4} - \frac{1}{2}\frac{2}{(s+2)^2+4}\right\} = e^{-2t}\cos 2t - \frac{1}{2}e^{-2t}\sin 2t$$

3. $\boxed{\text{25 Points}}$ Solve the following initial value problem by using Laplace Transformation.

$$y'' + y' - 6y = 50\cos t, \quad y(0) = 0, y'(0) = 0$$

Solution:

$$\begin{aligned}
\mathscr{L}\left\{y''\right\} &= s^2\mathscr{L}\{y\} - sy(0) - y'(0) = s^2\mathscr{L}\{y\} \\
\mathscr{L}\left\{y'\right\} &= s\mathscr{L}\{y\} - y(0) = s\mathscr{L}\{y\} \\
\mathscr{L}\{\cos t\} &= \frac{s}{s^2+1}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}\left\{y'' + y' - 6y\right\} &= \mathscr{L}\{50\cos t\} \\
s^2\mathscr{L}\{y\} + s\mathscr{L}\{y\} - 6\mathscr{L}\{y\} &= 50\mathscr{L}\{\cos t\} \\
(s^2+s-6)\mathscr{L}\{y\} &= \frac{50s}{s^2+1} \\
\mathscr{L}\{y\} &= \frac{50s}{(s^2+1)(s^2+s-6)} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{50s}{(s^2+1)(s^2+s-6)}\right\} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{50s}{(s^2+3)(s+3)(s-2)}\right\} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{4}{s-2} + \frac{3}{s+3} + \frac{1}{s^2+1} - \frac{7s}{s^2+1}\right\} \\
y(t) &= 4e^{2t} + 3e^{-3t} + \sin t - 7\cos t
\end{aligned}$$

4. $\boxed{\text{25 Points}}$ Find the solution of the following system of differential equations.

$$\mathbf{x}' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ -4 \end{bmatrix}.$$

Solution: Let us find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 1.2 = \lambda^2 - 7\lambda + 10 = (\lambda-5)(\lambda-2) = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 2.$$

The eigenvector corresponding to $\lambda_1 = 5$ is obtained by solving the system $(A - 5I)\mathbf{v} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 3-5 & 2 & 0 \\ 1 & 4-5 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvector corresponding to $\lambda_2 = 2$ is obtained by solving the system $(A - 2I)\mathbf{w} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 3-2 & 2 & 0 \\ 1 & 4-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The general solution of the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}.$$

To find particular solution, we use the initial values $x(0) = 5$ and $y(0) = -4$.

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \Rightarrow c_1 = -1 \quad \text{and} \quad c_2 = -3$$

Therefore

$$\mathbf{x}(t) = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} - 3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}$$

$$\mathbf{x}(t) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{5t} + \begin{bmatrix} 6 \\ -3 \end{bmatrix} e^{2t}.$$