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- A student who has cheated or attempted to cheat in the exam will get a **zero (0)**.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Elementary Laplace Transforms: Suppose that $a,b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathscr{L}\{f(t)\}$ exists and $F(s) = \mathscr{L}\{f(t)\}$

- $\mathscr{L}\{1\} = \frac{1}{s}, s > 0$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a},\ s > a,$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},\ s > 0,$
- $\mathscr{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$
- $\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$
- $\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}, s > 0$
- $\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2},\ s > 0$
- $\mathscr{L}\{\cosh at\} = \frac{s}{s^2-a^2},\ s > |a|$
- $\mathscr{L}\{\sinh at\} = \frac{a}{s^2-a^2},\ s > |a|$
- $\mathscr{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c}),\ c > 0$
- $\mathscr{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathscr{L}\{f(t)\}$
- $\mathscr{L}\{u_c(t)\} = \frac{e^{-cs}}{s},\ s > 0$
- $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

1. (a) 20 points Find the solution of the $2\frac{dy}{dt} - y = 2t$, $y(0) = y_0$, $y_0 \in \mathbb{R}$

Solution: $2\frac{dy}{dt} - y = 2t$ is a linear differential equation. Let us find the integrating factor.

$$\lambda = e^{\int -\frac{1}{2}dt} = e^{-\frac{t}{2}}$$

The general solution is

$$y(t) = e^{\frac{t}{2}} \int t e^{-\frac{t}{2}} dt$$
$$y(t) = e^{\frac{t}{2}} \left(-2te^{-\frac{t}{2}} + 2\int e^{-\frac{t}{2}} dt\right)$$
$$y(t) = e^{\frac{t}{2}} \left(-2te^{-\frac{t}{2}} - 4e^{-\frac{t}{2}} + C\right)$$
$$y(t) = -2t - 4 + Ce^{\frac{t}{2}}$$

Let us use initial values $y(0) = y_0$, $y(0) = -4 + C = y_0 \Rightarrow C = y_0 + 4$, then $y(t) = -2t - 4 + (y_0+4)e^{\frac{t}{2}}$

(b) 5 points There exists a number $a \in \mathbb{R}$ such that:
- If $y_0 < a$ then $y(t) \to -\infty$ as $t \to \infty$.
- If $y_0 > a$ then $y(t) \to \infty$ as $t \to \infty$. Find a.

Solution: If $y_0 + 4 \leq 0$ and $t \to \infty$, then $(y_0+4)e^{\frac{t}{2}} \to -\infty$ so $y(t) \to -\infty$.
If $y_0 + 4 > 0$ and $t \to \infty$, then $(y_0+4)e^{\frac{t}{2}} \to \infty$, so $y(t) \to \infty$.

2. (a) 15 points Find the general solution of $y' = \dfrac{2x-3y}{3x+5y}$.

Solution: First Way : Let us substitute $x \to \lambda x$ and $y \to \lambda y$, then

$$y' = \frac{2\lambda x - 3\lambda y}{3\lambda x + 5\lambda y} = \frac{\lambda(2x-3y)}{\lambda(3x+5y)} = \frac{2x-3y}{3x+5y}$$

Therefore, it is a homogeneous differential equation and we use the substitution $y = vx$.

$$\begin{aligned}
y = vx \Rightarrow \frac{dy}{dx} &= x\frac{dv}{dx} + v \\
x\frac{dv}{dx} + v &= \frac{2x-3vx}{3x+5vx} = \frac{2-3v}{3+5v} \\
x\frac{dv}{dx} = \frac{2-3v}{3+5v} - v &= \frac{2-3v-3v-5v^2}{3+5v} = \frac{2-6v-5v^2}{3+5v} \\
x\frac{dv}{dx} &= \frac{2-6v-5v^2}{3+5v} \\
\frac{3+5v}{2-6v-5v^2}dv &= \frac{dx}{x} \\
\int \frac{3+5v}{2-6v-5v^2}dv &= \int \frac{dx}{x} \\
-\frac{1}{2}\ln|2-6v-5v^2| &= lnx + lnC \\
2-6v-5v^2 &= \frac{1}{(Cx)^2} \\
2-6\frac{y}{x} - 5\frac{y^2}{x^2} &= \frac{1}{(Cx)^2} \\
2x^2-6xy-5y^2 = D, D &= \frac{1}{C^2}
\end{aligned}$$

Second Way: Verilen diferansiyel denklemi $(3y-2x)dx + (3x+5y)dy = 0$ şeklinde yeniden düzenleyelim.$M(x,y) = 3y-2x$ ve $N(x,y) = 3x+5y$ alalım. $M_y = N_x = 3$ olduğundan verilen denklem tam diferansiyel denklemdir. Buna göre $F_x(x,y)dx + F_y(x,y)dy = 0$ olacak şekilde bir $F(x,y) = 0$ fonksiyonu vardır. Dolayısıyla

$$\begin{aligned}
F_x = M(x,y) = 3y-2x \Rightarrow F(x,y) = \int (3y-2x)dx &\Rightarrow F(x,y) = 3xy - x^2 + h(y) \\
F_y = N(x,y) = 3x+5y \Rightarrow F_y = 3x + h'(y) = 3x+5y &\Rightarrow h(y) = \frac{5y^2}{2} + C \\
F(x,y) = 3xy - x^2 + \frac{5y^2}{2} + C &= 0
\end{aligned}$$

(b) 10 points Find the solution of the $y'' - 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 3$.

Solution: The characteristic equation of the given differential equation and its roots are obtained as follows.

$$\begin{aligned}
r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 &\Rightarrow r_1 = r_2 = 3 \\
y(x) &= c_1e^{3x} + c_2xe^{3x}
\end{aligned}$$

The solution of the initial value problem

$$\begin{aligned}
y(0) = 0 &\Rightarrow y(0) = c_1 = 0 \\
y'(0) = 3 \Rightarrow y' = 3c_1e^{3x} + 3c_2xe^{3x} + c_2e^{3x} \Rightarrow y'(0) = 3c_1 + c_2 = 3 &\Rightarrow c_2 = 3 \\
y(t) &= 3xe^{3x}
\end{aligned}$$

3. 25 points Solve the initial value problem $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Solution: First Way : The eigenvalue and the corresponding eigenvectors of matrix $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ are obtained by as follows

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-1-\lambda) - 1(-4) = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda-1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$$

$$\lambda_1 = \lambda_2 = 1 \Rightarrow (A-I)\mathbf{v} = \mathbf{0} \Rightarrow \left[\begin{array}{cc|c} 3-1 & 1 & 0 \\ -4 & -1-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(A-I)\mathbf{w} = \mathbf{v} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ 1-2w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} w_1 \Rightarrow w_1 = 0 \Rightarrow \mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The general solution of the given system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t$$

Let us find arbitrary coefficients c_1 and c_2 by using initial value $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow c_1 = 2 \text{ and } c_2 = 3$$

Therefore, the solution of the initial value problem is

$$\mathbf{x}(t) = 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t + 3 \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t$$

$$\mathbf{x}(t) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^t + \left(\begin{bmatrix} 3 \\ -6 \end{bmatrix} t + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) e^t$$

$$\mathbf{x}(t) = \begin{bmatrix} 3t+2 \\ -6t-1 \end{bmatrix} e^t$$

Alternatively we can find the solution by using fundamental matrix. The general solution can be rewrite as

$$\mathbf{x} = \begin{bmatrix} e^t & te^t \\ -2e^t & (-2t+1)e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \Psi(t) = \begin{bmatrix} e^t & te^t \\ -2e^t & (-2t+1)e^t \end{bmatrix} \Rightarrow \Psi^{-1}(t) = \frac{1}{e^{2t}} \begin{bmatrix} (-2t+1)e^t & -te^t \\ 2e^t & e^t \end{bmatrix} \Rightarrow \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{x}(t) = \Psi(t)\Psi^{-1}(0)\mathbf{x}(0) = \begin{bmatrix} e^t & te^t \\ -2e^t & (-2t+1)e^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ -2e^t & (-2t+1)e^t \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3t+2 \\ -6t-1 \end{bmatrix} e^t$$

Second Way : We can solve this problem by using Laplace Transformation.

$$\mathscr{L}\{\mathbf{x}'\} = \mathscr{L}\left\{\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\mathbf{x}\right\}$$

$$s\mathscr{L}\{\mathbf{x}\} - \mathbf{x}(0) = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\mathscr{L}\{\mathbf{x}\}$$

$$\left(sI - \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\right)\mathscr{L}\{\mathbf{x}\} = \mathbf{x}(0)$$

$$\begin{bmatrix} s-3 & -1 \\ 4 & s+1 \end{bmatrix}\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} s-3 & -1 \\ 4 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{(s-3)(s+1)-4(-1)}\begin{bmatrix} s+1 & 1 \\ -4 & s-3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\mathscr{L}\{\mathbf{x}\} = \frac{1}{s^2-2s+1}\begin{bmatrix} 2(s+1)+1(-1) \\ (-4)2+(s-3)(-1) \end{bmatrix} = \frac{1}{s^2-2s+1}\begin{bmatrix} 2s+1 \\ -s-5 \end{bmatrix}$$

$$\mathbf{x}(t) = \mathscr{L}^{-1}\left\{\begin{bmatrix} \dfrac{2s+1}{(s-1)^2} \\ \dfrac{-s-5}{(s-1)^2} \end{bmatrix}\right\}$$

$$\frac{2s+1}{(s-1)^2} = \frac{2(s-1)+3}{(s-1)^2} = \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\frac{-s-5}{(s-1)^2} = \frac{-(s-1)-6}{(s-1)^2} = -\frac{1}{s-1} - \frac{6}{(s-1)^2}$$

$$\Rightarrow \mathbf{x}(t) = \mathscr{L}^{-1}\left\{\begin{bmatrix} \dfrac{2}{s-1} + \dfrac{3}{(s-1)^2} \\ -\dfrac{1}{s-1} - \dfrac{6}{(s-1)^2} \end{bmatrix}\right\}$$

$$\mathbf{x}(t) = \begin{bmatrix} 2e^t + 3te^t \\ -e^t - 6te^t \end{bmatrix}$$

4. 25 points Find the solution of the initial value problem $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution: First Way : Let us find the eigenvalues and the corresponding eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - 3(-1) = \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -1$$

$$\lambda_1 = 1 \Rightarrow (A-I)\mathbf{v} = \mathbf{0} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\lambda_1 = -1 \Rightarrow (A+I)\mathbf{w} = \mathbf{0} \Rightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \Rightarrow \Psi^{-1}(t) = \frac{1}{-3+1}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{bmatrix} \Rightarrow \Psi^{-1}(0) = -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$e^{At} = \Psi(t)\Psi^{-1}(0) = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}\left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-Ax} f(x)\,dx$$

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{bmatrix}\int_0^t \frac{1}{2}\begin{bmatrix} 3e^{-x} - e^{x} & 3e^{-x} - 3e^{x} \\ -e^{-x} + e^{x} & -e^{-x} + 3e^{x} \end{bmatrix}\begin{bmatrix} e^{2x} \\ -e^{2x} \end{bmatrix} dx$$

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{bmatrix}\int_0^t \frac{1}{2}\begin{bmatrix} 3e^{x} - e^{3x} - 3e^{x} + 3e^{3x} \\ -e^{x} + e^{3x} + e^{x} - 3e^{3x} \end{bmatrix} dx$$

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{bmatrix}\int_0^t \begin{bmatrix} e^{3x} \\ -e^{3x} \end{bmatrix} dx$$

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{bmatrix}\left(\begin{bmatrix} \frac{e^{3t}}{3} \\ -\frac{e^{3t}}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}\right)$$

$$\mathbf{x}(t) = \frac{1}{6}\left(\begin{bmatrix} 3e^{4t} - e^{2t} - 3e^{4t} + 3e^{2t} \\ -e^{4t} + e^{2t} + e^{4t} - 3e^{2t} \end{bmatrix} + \begin{bmatrix} -3e^t + e^{-t} + 3e^t - 3e^{-t} \\ e^t - e^{-t} - e^t + 3e^{-t} \end{bmatrix}\right)$$

$$x(t) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \frac{1}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Second Way : Let us find the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - 3(-1) = \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -1$$

$$\lambda_1 = 1 \Rightarrow (A-I)\mathbf{v} = \mathbf{0} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\lambda_1 = -1 \Rightarrow (A+I)\mathbf{w} = \mathbf{0} \Rightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_h(t) = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Let us find the $\mathbf{x}_p$ by using Method of Undetermined Coefficients.

$$\mathbf{x}_p = \begin{bmatrix} A \\ B \end{bmatrix} e^{2t} \Rightarrow \mathbf{x}'_p = \begin{bmatrix} 2A \\ 2B \end{bmatrix} e^{2t}$$

$$\mathbf{x}'_p = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \mathbf{x}_p + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$\begin{bmatrix} 2A \\ 2B \end{bmatrix} e^{2t} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} 2A+3B+1 \\ -A-2B-1 \end{bmatrix} e^{2t}$$

$$\Rightarrow \begin{array}{l} 2A = 2A+3B+1 \\ 2B = -A-2B-1 \end{array} \Rightarrow A = \frac{1}{3} B = -\frac{1}{3}$$

$$\mathbf{x}_p = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

We can use method of Variation of parameters to find $\mathbf{x}_p$.

$$\mathbf{x}_p = \Psi(t) \int \Psi^{-1}(t)\mathbf{f}(t)dt \Rightarrow \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \Rightarrow \Psi^{-1}(t) = -\frac{1}{2} \begin{bmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{bmatrix}$$

$$\mathbf{x}_p = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int -\frac{1}{2} \begin{bmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{bmatrix} \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} dt$$

$$\mathbf{x}_p = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} 0 \\ -e^{3t} \end{bmatrix} dt$$

$$\mathbf{x}_p = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{3} e^{3t} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

The general solution is $\mathbf{x}(t) = \mathbf{x}_h + \mathbf{x}_p = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$. Let us use initial values to find c_1 and c_2.

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}(0) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} -3c_1 - c_2 + \frac{1}{3} = 0 \\ c_1 + c_2 - \frac{1}{3} = 0 \end{array} \Rightarrow c_1 = 0, c_2 = \frac{1}{3}$$

The solution of the initial value problem is $\mathbf{x}(t) = \mathbf{x}_h + \mathbf{x}_p = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$.

Third Way : Let us solve the system by using Laplace Transformation.

$$\mathscr{L}\{\mathbf{x}'\} = \mathscr{L}\left\{\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}\right\}$$

$$s\mathscr{L}\{\mathbf{x}\} - \mathbf{x}(0) = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\mathscr{L}\{\mathbf{x}\} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\mathscr{L}\{e^{2t}\}$$

$$\left(sI - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\right)\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{s-2}$$

$$\begin{bmatrix} s-2 & -3 \\ 1 & s+2 \end{bmatrix}\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{s-2}$$

$$\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} s-2 & -3 \\ 1 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{s-2} \\ -\frac{1}{s-2} \end{bmatrix} = \frac{1}{(s-2)(s+2)+3}\begin{bmatrix} s+2 & 3 \\ -1 & s-2 \end{bmatrix}\begin{bmatrix} \frac{1}{s-2} \\ -\frac{1}{s-2} \end{bmatrix} = \frac{1}{s^2-1}\begin{bmatrix} \frac{s-1}{s-2} \\ \frac{-s+1}{s-2} \end{bmatrix}$$

$$\frac{s-1}{(s-2)(s-1)(s+1)} = \frac{1}{(s-2)(s+1)} = \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}$$

$$\frac{-s+1}{(s-2)(s-1)(s+1)} = -\frac{1}{(s-2)(s+1)} = -\frac{1}{3}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s+1}$$

$$\mathbf{x}(t) = \mathscr{L}^{-1}\left\{\begin{bmatrix} \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1} \\ -\frac{1}{3}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s+1} \end{bmatrix}\right\}$$

$$\mathbf{x}(t) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \frac{1}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$