Your Name / İsim Soyisim

Your Signature / İmza

Student ID # / Öğrenci Numarası

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- A student who has cheated or attempted to cheat in the exam will get a **zero (0)**.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 75 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. $\boxed{\text{20 points}}$ Solve the initial value problem $ty' + 2y = t^2 - t + 1,\ y(1) = 2,\ t > 0.$

Solution: $ty' + 2y = t^2 - t + 1$ is a linear differential equation. We can arrange the equation $y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$ and the integrating factor is

$$\lambda(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

Let us multiply the equation by the integrating factor.

$$\begin{aligned} t^2y' + 2ty &= t^3 - t^2 + t \\ \frac{d}{dt}(t^2y) &= t^3 - t^2 + t \\ t^2y = \int (t^3 - t^2 + t)dt &= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C \\ y(t) &= \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2} \end{aligned}$$

Let us use the initial value to find arbitrary constant.

$$y(1) = 2 \Rightarrow y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = 2 \Rightarrow C = \frac{19}{12}$$

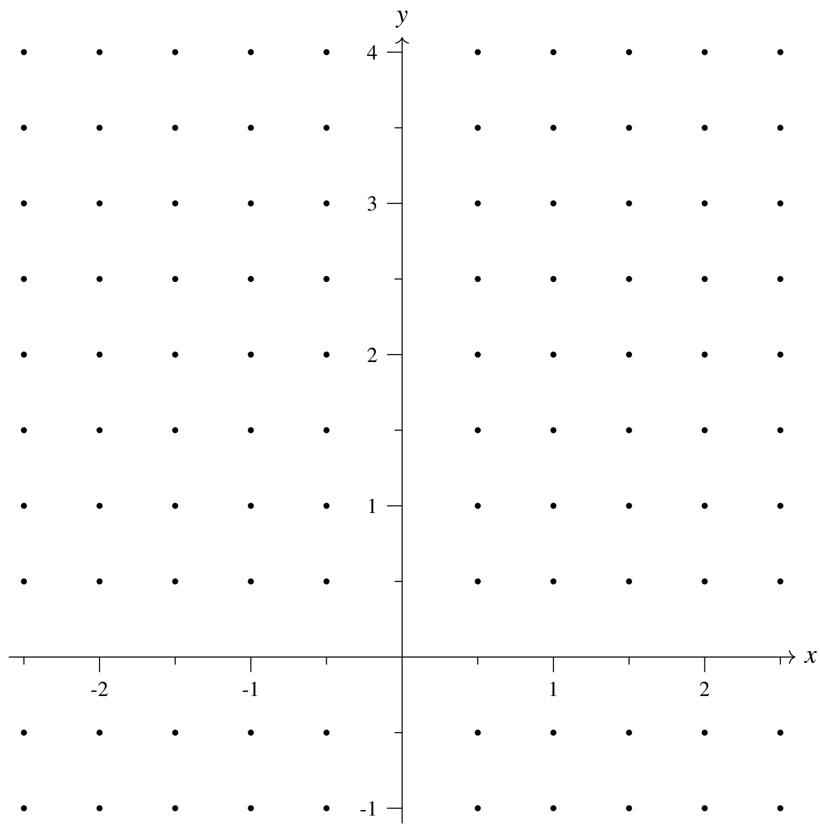
The solution of the initial value problem is $y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{19}{12t^2}$

2. (a) $\boxed{\text{5 points}}$ Classify the following differential equation.Write the order, linearity and the homogeneity of the equation.

$$\frac{d^3y}{dx^3}+2e^x\frac{d^2y}{dx^2}=x^3+5xy.$$

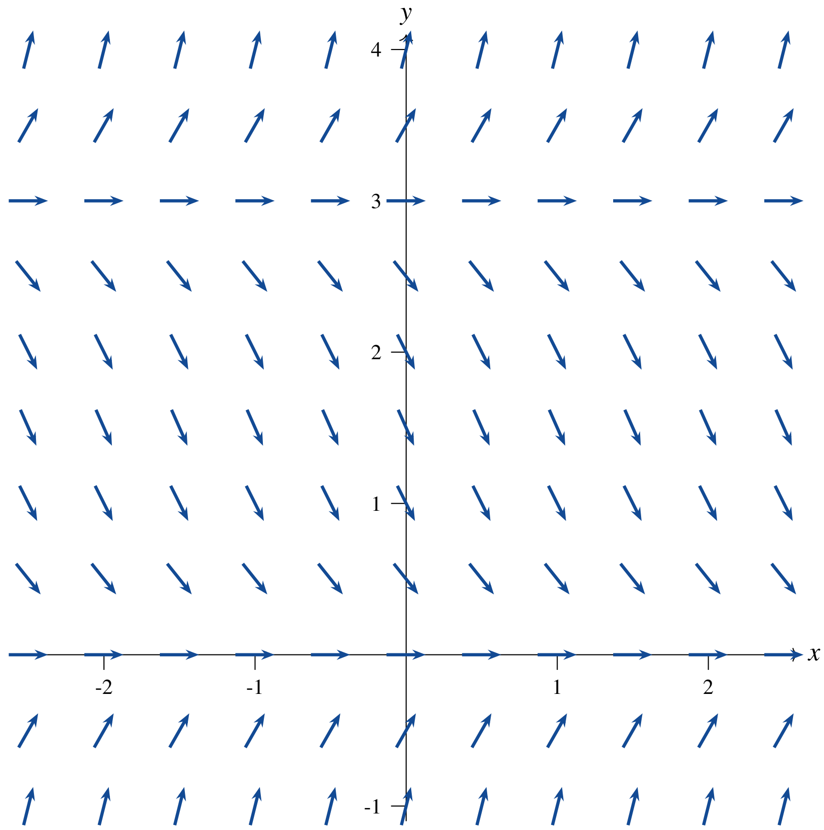
Solution: The given equation is a third order, linear, and non-homegeneous differential equation.

(b) $\boxed{\text{15 points}}$ Draw a direction field for $y'=-y(3-y)$.



Solution: $y'=0$ at the points $y=0$ and $y=3$.

If $0<y<3,$ then $y'<0$ and y is a decreasing function.
If $y<0$ and $3<y,$ then$y'>0$and y is an increasing function.


y
4
3
2
1
-1
x
-2
-1
1
2

3. 20 points Find the general solution of $(4xy^2+4y)dx+(4x^2y+4x)dy=0$.

Solution: Let us take $M(x,y)=4xy^2+4y$ and $N(x,y)=4x^2y+4x$. Let us calculate $M_y=8xy+4=N_x$, so the given equation is an exact differential eqution. There exists a function $F(x,y)=0$ such that $F_xdx+F_ydy=0$. Therefore $F_x=4xy^2+4y$ ve $F_y=4x^2y+4x$.

$$F_x=4xy^2+4y \Rightarrow F(x,y)=\int(4xy^2+4y)dx=2x^2y^2+4xy+h(y)$$
$$F_y=4x^2y+4x \Rightarrow F_y=4x^2y+4x+h'(y)=4x^2y+4x \Rightarrow h'(y)=0 \Rightarrow h(y)=C$$
$$F(x,y)=2x^2y^2+4xy+C=0$$

4. Suppose that $(r-1)(r^2+9)=0$ is the characteristic equation of a linear, homogeneous and constant coefficient differential equation.
   (a) 5 points Determine the differential equation mentioned above.
   (b) 15 points Find the solution of the differential equation, that you wrote in part (a), which satisfies the initial conditions $y(0)=2, y'(0)=-3, y''(0)=12$.

Solution:

(a) $(r-1)(r^2+9)=0 \Rightarrow r^3-r^2+9r-9=0 \Rightarrow \dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}+9\dfrac{dy}{dx}-9y=0$

(b) $(r-1)(r^2+9)=0 \Rightarrow r_1=1,\ r_2=3i,\ r_3=-3i$ and the general solution of the given differential equation is $y(x)=c_1e^x+c_2\cos 3x+c_3\sin 3x$.
Let us use the initial values to find arbitrary coefficients.

$$y(0)=2 \Rightarrow y(0)=c_1e^0+c_2\cos 0+c_3\sin 0=2 \Rightarrow c_1+c_2=2$$
$$y'(0)=-3 \Rightarrow y'(x)=c_1e^x-3c_2\sin 3x+3c_3\cos 3x \Rightarrow y'(0)=c_1e^0-3c_2\sin 0+3c_3\cos 0 \Rightarrow c_1+3c_3=-3$$
$$y''(0)=12 \Rightarrow y''(x)=c_1e^x-9c_2\cos 3x-9c_3\sin 3x \Rightarrow y''(0)=c_1e^0-9c_2\cos 0-9c_3\sin 0=12 \Rightarrow c_1-9c_2=12$$

when we solve the system, we obtain $c_1=3,\ c_2=-1,\ c_3=-2$. The particular solution of the given differential equation is $y(x)=3e^x-\cos 3x-2\sin 3x$.

5. $\boxed{\text{20 points}}$ Find the general solution of $y'' - y' - 2y = -3 + 4t^2$.

Solution: Let us find the general solution of the $y'' - y' - 2y = 0$. The characteristic equation and its roots are

$$r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$$

and the gereral solution is $y_h(t) = c_1 e^{2t} + c_2 e^{-t}$.
We can determine the particular solution by using method of undetermined coefficient, so $y_p(t) = At^2 + Bt + C$.The derivatives of the y_p are $y_p' = 2At + B$ and $y_p'' = 2A$.

$$\begin{aligned} y_p'' - y_p' - 2y_p &= -3 + 4t^2 \\ 2A - (2At + B) - 2(At^2 + Bt + C) &= -3 + 4t^2 \\ -2At^2 + (-2A - 2B)t + (2A - B - 2C) &= -3 + 4t^2 \\ \Rightarrow A = -2, B = 2, C &= -\frac{3}{2} \\ y_p(t) &= -2t^2 + 2t - \frac{3}{2} \end{aligned}$$

The general solution of the given differential equation is $y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^{-t} - 2t^2 + 2t - \dfrac{3}{2}$.