Cep telefonunuzu gözetmene teslim ediniz / Deposit your cell phones to invigilator

April 27, 2017 [16:00-17:10] MATH216 Second Midterm Exam / MAT216 İkinci Ara Sınav

Your Name / İsim Soyisim

Your Signature / İmza

Student ID # / Öğrenci Numarası

Professor's Name / Öğretim Üyesi

- A student who has cheated or attempted to cheat in the exam will get a **zero (0)**.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

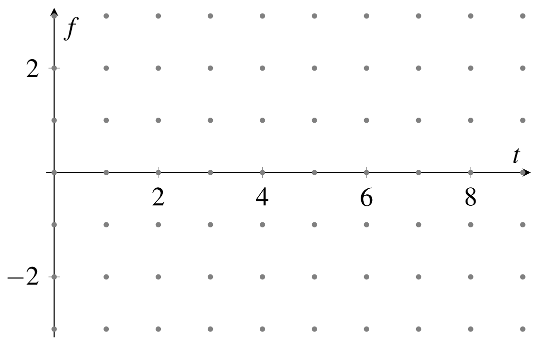
Elementary Laplace Transforms: Suppose that $a,b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathscr{L}\{f(t)\}$ exists and $F(s) = \mathscr{L}\{f(t)\}$

- $\mathscr{L}\{1\} = \dfrac{1}{s}, s > 0$
- $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a},\ s > a,$
- $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\ s > 0,$
- $\mathscr{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$
- $\mathscr{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$
- $\mathscr{L}\{\cos at\} = \dfrac{s}{s^2+a^2}, s > 0$
- $\mathscr{L}\{\sin at\} = \dfrac{a}{s^2+a^2},\ s > 0$
- $\mathscr{L}\{\cosh at\} = \dfrac{s}{s^2-a^2},\ s > |a|$
- $\mathscr{L}\{\sinh at\} = \dfrac{a}{s^2-a^2},\ s > |a|$
- $\mathscr{L}\{f(ct)\} = \dfrac{1}{c}F(\dfrac{s}{c}),\ c > 0$
- $\mathscr{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathscr{L}\{f(t)\}$
- $\mathscr{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s},\ s > 0$
- $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathscr{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

1. (a) 5 points Find the Laplace Transform of $f(t) = 2 - t^3 + 4\sin 5t - 2e^{4t}$.

Solution: $\mathscr{L}\{f(t)\} = \mathscr{L}\left\{2 - t^3 + 4\sin 5t - 2e^{4t}\right\} = \dfrac{2}{s} - \dfrac{3!}{s^4} + 4\dfrac{5}{s^2+25} - 2\dfrac{1}{s-4}, s > 4$

(b) 20 points Suppose that $f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -1, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & 7 \le t \end{cases}$.



(5 points) Sketch the graph of $f(t)$

(5 points) Express $f(t)$ in terms of the unit step function $u_c(t)$.

(10 points) Find the Laplace Transform of $f(t)$.

Solution: We express $f(t)$ in terms of unit step functions $u_c(t)$.

$$f(t) = -u_3(t) + 3u_5(t) - u_7(t)$$

Let us calculate the Laplace transform of $f(t)$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{-u_3(t) + 3u_5(t) - u_7(t)\} = -\frac{e^{-3s}}{s} + \frac{2e^{-5s}}{s} - \frac{e^{-7s}}{s}$$

2. (a) 10 points Find the inverse Laplace Transform of $F(s) = \dfrac{2s+3}{s^2-2s+2}$.

Solution:

$$f(t) = \mathscr{L}^{-1}\left\{\frac{2s+3}{s^2-2s+2}\right\} = \mathscr{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+1} + 5\frac{1}{(s-1)^2+1}\right\} = 2e^{-t}\cos t + 5e^{-t}\sin t$$

(b) 15 points Find the inverse Laplace Transform of $F(s) = \dfrac{s^2+1}{(s+1)(s+2)(s-3)}$.

Solution:

$$f(t) = \mathscr{L}^{-1}\left\{\frac{s^2+1}{(s+1)(s+2)(s-3)}\right\} = \mathscr{L}^{-1}\left\{\frac{A}{S+1} + \frac{B}{s+2} + \frac{C}{s-3}\right\}$$

$$\Rightarrow s^2+1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2) \Rightarrow A = -\frac{1}{2}, B = 1, C = \frac{1}{2}$$

$$\Rightarrow f(t) = \mathscr{L}^{-1}\left\{-\frac{1}{2}\frac{1}{S+1} + \frac{1}{s+2} + \frac{1}{2}\frac{1}{s-3}\right\}$$

$$f(t) = -\frac{1}{2}e^{-t} + e^{-2t} + \frac{1}{2}e^{3t}$$

3. 25 points Use the Laplace Transform to solve the initial value problem $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = -1$.

Solution: Let us calculate the Laplace Transform of the given differential equation.

$$\begin{aligned}
\mathscr{L}\left\{y'' + 2y' + y\right\} &= \mathscr{L}\left\{4e^{-t}\right\} \\
\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] + 2\left[s\mathscr{L}\{y\} - y(0)\right] + \mathscr{L}\{y\} &= \frac{4}{s+1} \\
(s^2 + 2s + 1)\mathscr{L}\{y\} - 2s + 1 - 4 &= \frac{4}{s+1} \\
(s+1)^2\mathscr{L}\{y\} &= \frac{4}{s+1} + 2s + 3 \\
\mathscr{L}\{y\} &= \frac{2s^2 + 5s + 7}{(s+1)^3} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{2s^2 + 5s + 7}{(s+1)^3}\right\}
\end{aligned}$$

Let us write

$$\begin{aligned}
\frac{2s^2 + 5s + 7}{(s+1)^3} &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} \\
\Rightarrow 2s^2 + 5s + 7 = A(s+1)^2 + B(s+1) + Cs &= -1 \Rightarrow 4 = C \\
\Rightarrow 2s^2 + 5s + 3 = A(s+1)^2 + B(s+1) \Rightarrow (s+1)(s+3) = (s+1)\left[A(s+1) + B\right] \Rightarrow s &= -1 \Rightarrow B = 2 \text{ and } A = 1 \\
\frac{2s^2 + 5s + 7}{(s+1)^3} &= \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{4}{(s+1)^3} \\
y(t) = \mathscr{L}^{-1}\left\{\frac{2s^2 + 5s + 7}{(s+1)^3}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{4}{(s+1)^3}\right\} \\
y(t) &= e^{-t} + 2te^{-t} + 2t^2e{-}t
\end{aligned}$$

4. 25 points Use the Laplace Transform to find the solution of the initial value problem $y'' + 9y = f(t)$, $y(0) = 0$, $y'(0) = 1$ where $f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t \end{cases}$.

Solution: The Laplace Transform of $f(t) = 1 - u_{3\pi}(t)$ is $F(s) = \frac{1}{s} - \frac{e^{-3\pi s}}{s}$. Therefore

$$\begin{aligned} \mathscr{L}\left\{y'' + 9y\right\} &= \mathscr{L}\left\{f(t)\right\} \\ (s^2+9)\mathscr{L}\left\{y\right\} - 1 &= \frac{1 - e^{-3\pi s}}{s} \\ \mathscr{L}\left\{y\right\} &= \frac{s + 1 - e^{-3\pi s}}{s(s^2+9)} \\ \mathscr{L}\left\{y\right\} &= \frac{s+1}{s(s^2+9)} - \frac{e^{-3\pi s}}{s(s^2+9)} \end{aligned}$$

$$\begin{aligned} \frac{s+1}{s(s^2+9)} &= \frac{A_1}{s} + \frac{B_1 s + C_1}{s^2+9} \\ s + 1 = A_1(s^2+9) + (B_1 s + C_1)s \Rightarrow s + 1 = (A_1 + B_1)s^2 + C_1 s + 9A_1 \Rightarrow A_1 &= \frac{1}{9},\ B_1 = -\frac{1}{9},\ C_1 = 1 \\ \frac{1}{s(s^2+9)} &= \frac{A_2}{s} + \frac{B_2 s + C_2}{s^2+9} \\ 1 = A_2(s^2+9) + (B_2 s + C_2)s \Rightarrow 1 = (A_2 + B_2)s^2 + C_2 s + 9A_2 \Rightarrow A_2 &= \frac{1}{9},\ B_2 = -\frac{1}{9},\ C_2 = 0 \\ y(t) = \mathscr{L}^{-1}\left\{\frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+9}\right\} &- \mathscr{L}^{-1}\left\{\frac{1}{9}\frac{e^{-3\pi s}}{s} - \frac{1}{9}\frac{se^{-3\pi s}}{s^2+9}\right\} \\ y(t) = \frac{1}{9} - \frac{1}{9}\cos 3t + \frac{1}{3}\sin 3t &- \frac{1}{9}u_{3\pi}(t) + \frac{1}{9}u_{3\pi}(t)\cos(3(t-3\pi)) \\ y(t) = \frac{1}{9} - \frac{1}{9}\cos 3t + \frac{1}{3}\sin 3t &- \frac{1}{9}u_{3\pi}(t) - \frac{1}{9}u_{3\pi}(t)\cos(3t) \end{aligned}$$