FORENAME:

SURNAME:

STUDENT NO:

DEPARTMENT:

TEACHER: ☐ Neil Course ☐ Vasfi Eldem ☐ M.Tuba Gülpınar ☐ Hasan Özekes

SIGNATURE:

Question	Points	Score
1	20	
2	30	
3	25	
4	25	
Total:	100	

- The time limit is 75 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. $\boxed{\text{20 points}}$ Find the solution of the initial value problem $\dfrac{dy}{dx} = \dfrac{x^2 - xy + y^2}{xy}$, $y(3) = 0$ by using the substitution $y = vx$.

Solution: Let us express the $\dfrac{dy}{dx} = \dfrac{x^2 - xy + y^2}{xy}$ as afunction of $\dfrac{y}{x}$.

$$\frac{1 - \frac{y}{x} + (\frac{y}{x})^2}{\frac{y}{x}}$$

It is a homogeneous equation and we use the substitution $v = \frac{y}{x}$ to solve it.

$$v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy} \Rightarrow x\frac{dv}{dx} + v = \frac{x^2 - x(vx) + (vx)^2}{x(vx)}$$

$$x\frac{dv}{dx} = \frac{1 - v + v^2}{v} - v = \frac{1-v}{v}$$

$$\frac{v}{v-1}dv = -\frac{1}{x}dx \Rightarrow \int \frac{v}{v-1}dv = -\int \frac{1}{x}dx$$

$$\int \left(1 + \frac{1}{v-1}\right)dv = -\int \frac{1}{x}dx$$

$$\Rightarrow v + \ln|v-1| = -\ln|x| + C \Rightarrow \frac{y}{x} + \ln\left|\frac{y}{x} - 1\right| = -\ln|x| + C$$

Let us substitute the initial value. $x = 3$ and $y = 0$.

$$y(3) = 0 \Rightarrow \frac{0}{3} + \ln\left|\frac{0}{3} - 1\right| = -\ln|3| + C \Rightarrow C = \ln 3$$

$$\frac{y}{x} + \ln\left|\frac{y}{x} - 1\right| = -\ln|x| + \ln 3$$

$$e^{\frac{y}{x}}\left(\frac{y}{x} - 1\right) = \frac{3}{x}$$

2. (a) $\boxed{\text{15 points}}$ Find the general solution of $\dfrac{dy}{dt} + \dfrac{3}{t}y = \dfrac{\cos t}{t^3}$ where $t > 0$.

Solution: It is a linear differential equation. Let us find the integrating factor.

$$\mu(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3$$

Calculate the product of $\mu(t)$ and the given differential equation.

$$t^3\frac{dy}{dt} + 3t^2y = \cos t$$

$$\frac{d}{dt}\left(t^3y\right) = \cos t \Rightarrow t^3y = \int \cos t dt \Rightarrow t^3y = \sin t + C \Rightarrow y(t) = \frac{\sin t + C}{t^3}$$

(b) $\boxed{\text{15 points}}$ i. Find the equilibrium solutions for $y' = y(4 - y^2)$.

ii. Sketch the direction field of $y' = y(4 - y^2)$ (You are expected to draw 121 arrows).

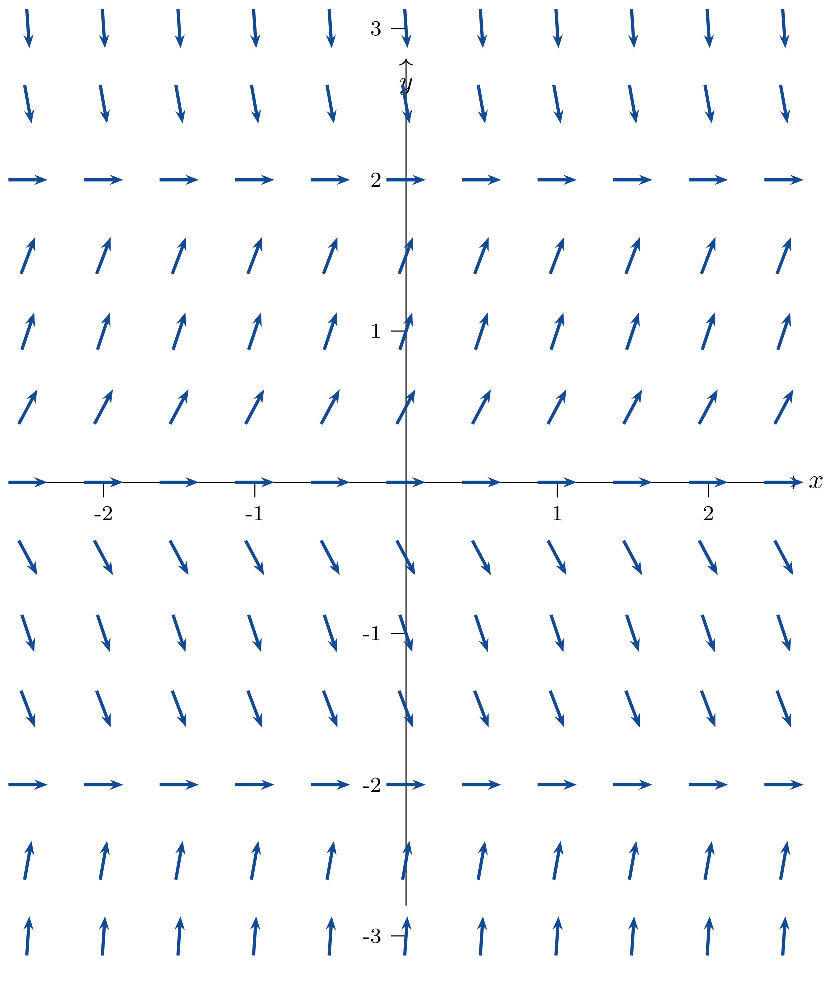
iii. Classify each equlibrium solution as asymptotically stable or unstable.

Solution:

(a) At the points $y = 0$, $y = 2$, and $y = -2$, we have $y' = 0$. Therefore $y = 0$, $y = 2$, and $y = -2$ are the equilibrium solution.

(b) Let us determine the sign of y'

if $-2 < y < 0$ or $y > 2$ then $y' < 0$ so, it is a decreasing function.
if $y < -2$ or $0 < y < 2$ then $y' > 0$ so, it is an increasing function.



$y = -2$ and $y = 2$ are asymptotically stable and, $y = 0$ is unstable.

3. $\boxed{\text{25 points}}$ Find the general solution of $y''' - 2y'' + y' = 12e^x + 5$.

Solution: Let us find the general solution of $y''' - 2y'' + y' = 0$. The characteristic equation is $r^3 - 2r^2 + r = 0$, and its roots are $r_1 = 0$ and $r_2 = r_3 = 1$.Therefore

$$y_h(x) = c_1 + c_2e^x + c_3xe^{3x}$$

We can find $y_p(x)$ by using method of undetermined .

$$\begin{aligned} y_p(x) &= Ax^2e^x + Bx \\ y_p'(x) &= 2Axe^x + Ax^2e^x + B \\ y_p''(x) &= 2Ae^x + 4Axe^x + Ax^2e^x \\ y_p'''(x) &= 6Ae^x + 6Axe^x + Ax^2e^x \end{aligned}$$

Let us substitute them

$$\begin{aligned} y''' - 2y'' + y' &= \left[6Ae^x + 6Axe^x + Ax^2e^x\right] - 2\left[2Ae^x + 4Axe^x + Ax^2e^x\right] + \left[2Axe^x + Ax^2e^x + B\right] \\ &= (6A - 4A)e^x + (6A - 8A + 2A)xe^x + (A - 2A + A)x^2e^x + B \\ &\Rightarrow y''' + 6y'' + 9y' = 12e^x + 5 \\ &2Ae^x + B = 12e^x + 5 \\ &\Rightarrow 2A = 12 \Rightarrow A = 6 \\ &\Rightarrow B = 5 \Rightarrow B = 5 \end{aligned}$$

Therefore $y_p(x) = 6x^2e^x + 5x$. The general solution of the equation is

$$y(x) = c_1 + c_2e^x + c_3xe^x + 6x^2e^x + 5x$$

4. $\boxed{\text{25 points}}$ Find the general solution of $y'' - 2y' + 2y = e^x \sec x$.

Solution: Let us find the general solution of $y'' - 2y' + 2y = 0$.The characteristik equation of the homogeneous equation is $r^2 - 2r + 2 = 0$ and its roots are $r_1 = 1 + i$ and $r_2 = 1 - i$. Therefore, the general solution of the homegeneous equation is

$$y_h(x) = c_1 e^x \cos x + c_2 e^x \sin x$$

Let us find y_p by using variation of parameters. $y_p(x) = u_1 e^x \cos x + u_2 e^x \sin x$ and we solve the system

$$\begin{aligned} u_1' e^x \cos x + u_2' e^x \sin x = 0 \\ u_1' e^x \cos x - u_1' e^x \sin x + u_2' e^x \sin x + u_2' e^x \cos x = e^x \sec x \\ \text{so} \\ u_1' e^x \cos x + u_2' e^x \sin x = 0 \\ -u_1' e^x \sin x + u_2' e^x \cos x = e^x \sec x \end{aligned}$$

When we solve the system, we obtain $u_1' = -\dfrac{\sin x}{\cos x}$ and $u_2' = 1$, so $u_1 = ln\,|\cos x|$ and $u_2 = x$.

$$y_p(x) = e^x \cos x ln\,|\cos x| + x e^x \sin x$$

The general solution of the differetial equation is $y(x) = y_h(x) + y_p(x)$, so

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x + e^x \cos x ln\,|\cos x| + x e^x \sin x$$