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DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	25	
2	20	
3	30	
4	25	
Total:	100	

- The time limit is 70 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

Elementary Laplace Transforms: Suppose that $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathcal{L}\{f(t)\}$ exists and $F(s) = \mathcal{L}\{f(t)\}$

- $\mathcal{L}\{1\} = \frac{1}{s}, s > 0$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \ s > a,$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \ s > 0,$
- $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$
- $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$
- $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, s > 0$
- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \ s > 0$
- $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
- $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c}), \ c > 0$
- $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \ s > 0$
- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

1. $\boxed{\text{25 points}}$ Find the Laplace Transform, $Y(s) = \mathcal{L}\{y(t)\}$, of the solution of the initial value problem $y'' - 4y' + 4y = t\sin t$, $y(0) = 1$, $y'(0) = -3$.

Solution:

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t\sin t\}$$
$$\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - 4\left[s\mathcal{L}\{y\} - y(0)\right] + 4\mathcal{L}\{y\} = \mathcal{L}\{t\sin t\}$$
$$\left[s^2\mathcal{L}\{y\} - s + 3\right] - 4\left[s\mathcal{L}\{y\} - 1\right] + 4\mathcal{L}\{y\} = \mathcal{L}\{t\sin t\}$$
$$(s^2 - 4s + 4)\mathcal{L}\{y\} - s + 7 = \frac{2s}{(s^2+1)^2} = \frac{2s}{s^4+2s^2+1}$$
$$(s^2 - 4s + 4)\mathcal{L}\{y\} = \frac{2s}{s^4+2s^2+1} + s - 7 = \frac{s^5 - 7s^4 + 2s^3 - 14s^2 + 3s - 7}{s^4+2s^2+1}$$
$$\mathcal{L}\{y\} = \frac{s^5 - 7s^4 + 2s^3 - 14s^2 + 3s - 7}{(s^4+2s^2+1)(s^2-2s+1)}$$

2. $\boxed{\text{20 points}}$ Find the inverse Laplace Transform of $F(s) = \dfrac{6s^2+10s+14}{(s-1)(s+2)(s^2+2s+2)}$.

Solution:

$$\frac{6s^2+10s+14}{(s-1)(s+2)(s^2+2s+2)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+2s+2}$$

$$\Rightarrow 6s^2+10s+14 = A(s+2)(s^2+2s+2) + B(s-1)(s^2+2s+2) + (Cs+D)(s-1)(s+2)$$

$$s=1 \Rightarrow 30 = 15A \Rightarrow A = 2$$

$$s=-2 \Rightarrow 18 = -6B \Rightarrow B = -3$$

$$s=0 \Rightarrow 14 = 4A - 2B - 2D \Rightarrow D = 0$$

$$s=-1 \Rightarrow 10 = A - 2B + 2C - 2D \Rightarrow C = 1$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6s^2+10s+14}{(s-1)(s+2)(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{3}{s+2} + \frac{s}{s^2+2s+2}\right\}$$

$$f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$f(t) = 2e^{t} - 3e^{-2t} + e^{-t}\cos t - e^{-t}\sin t$$

3. $\boxed{\text{30 points}}$ Find the solution of the initial value problem $y''+y=f(t)$, $y(0)=0$ and $y'(0)=0$ by using the Laplace Transform, where $f(t)=\begin{cases} t, & 0\le t<3, \\ 0, & 3\le t<\infty \end{cases}$.

Solution:

$$\begin{aligned} f(t) &= t - tu_3(t) \\ \Rightarrow \mathcal{L}\{f(t)\} &= \mathcal{L}\{t-tu_3(t)\} = \mathcal{L}\{t\} - e^{-3s}\mathcal{L}\{t+3\} \\ &= \frac{1}{s^2} - \frac{1}{s^2}e^{-3s} - \frac{3}{s}e^{-3s} = \frac{1-(3s+1)e^{-3s}}{s^2} \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{y''+y\} &= \mathcal{L}\{f(t)\} \\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} &= \frac{1-(3s+1)e^{-3s}}{s^2} \\ (s^2+1)\mathcal{L}\{y\} &= \frac{1-(3s+1)e^{-3s}}{s^2} \\ \mathcal{L}\{y\} &= \frac{1-(3s+1)e^{-3s}}{s^2(s^2+1)} \\ \Rightarrow y(t) &= \mathcal{L}^{-1}\left\{\frac{1-(3s+1)e^{-3s}}{s^2(s^2+1)}\right\} \\ \frac{1}{s^2(s^2+1)} &= \frac{A_1}{s} + \frac{B_1}{s^2} + \frac{C_1s+D_1}{s^2+1} \\ 1 = A_1s(s^2+1) + B_1(s^2+1) + (C_1s+D_1)s^2 &\Rightarrow A_1=0, B_1=1, C_1=0, D_1=-1 \\ \Rightarrow \frac{1}{s^2(s^2+1)} &= \frac{1}{s^2} - \frac{1}{s^2+1} \\ \frac{3s+1}{s^2(s^2+1)} &= \frac{A_2}{s} + \frac{B_2}{s^2} + \frac{C_2s+D_2}{s^2+1} \\ 3s+1 = A_2s(s^2+1) + B_2(s^2+1) + (C_2s+D_2)s^2 &\Rightarrow A_2=3, B_2=1, C_2=-3, D_2=-1 \\ \frac{3s+1}{s^2(s^2+1)} &= \frac{3}{s} + \frac{1}{s^2} + \frac{-3s-1}{s^2+1} \\ y(t) = \mathcal{L}^{-1}\left\{\frac{1-(3s+1)e^{-3s}}{s^2(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}(3s+1)}{s^2(s^2+1)}\right\} \\ y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} &- \mathcal{L}^{-1}\left\{\left(\frac{3}{s} + \frac{1}{s^2} + \frac{-3s-1}{s^2+1}\right)e^{-3s}\right\} \\ y(t) = (t-\sin t) - [3 + (t-3) &- 3\cos(t-3) - \sin(t-3)]\,u_3(t) \end{aligned}$$

4. $\boxed{\text{25 points}}$ Find the solution of the initial value problem $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix}$.

Hint: The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = -3$.

Solution: The eigenvalues of matrix A are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = -3$. Let us find the corresponding eigenvectors.

$$\lambda_1 = 1 \Rightarrow (A - I)\mathbf{u} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 2 & 2 & -4 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2 \Rightarrow (A - 2I)\mathbf{v} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 1 & -4 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -3 \Rightarrow (A + 3I)\mathbf{w} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 2 & 6 & -4 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -11 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 11 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

The general solution of the problem is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} e^{-3t}$$

We use the initial condition to determine the arbitrary coefficients.

$$\begin{array}{l} c_1 + c_2 + c_3 = 2 \\ c_1 + 2c_2 + 7c_3 = 6 \\ c_1 + c_2 + 11c_3 = 12 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 7 & 6 \\ 1 & 1 & 11 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 10 & 10 \end{array}\right] \Rightarrow c_1 = 3, c_2 = -2, c_3 = 1$$

The solution of the initial value problem is

$$\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t - 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} e^{-3t} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} e^t + \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} e^{-3t}$$