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DEPARTMENT:

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SIGNATURE:

| Question | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 25 | |
| Total: | 100 | |

- The time limit is 90 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- All communication between students, either verbally or non-verbally, is strictly forbidden.
- Calculators, mobile phones, smart watches, and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Please do not write in the table above.

1. Consider

$$y + (2x - 3ye^y)y' = 0. \tag{1}$$

(a) $\boxed{\text{2 points}}$ Show that (1) is not exact .

**Solution:** Let $M(x,y) = y$ and $N(x,y) = 2x - 3ye^y$. Since $M_y = 1 \neq 2 = N_x$, equation (1) is not exact.

(b) $\boxed{\text{10 points}}$ Find an integrating factor $\mu(y)$ which can be used to convert (1) into an exact equation.

**Solution:** Solving

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = \left(\frac{2-1}{y}\right)\mu = \frac{\mu}{y} \quad \Longrightarrow \quad \int \frac{d\mu}{\mu} = \int \frac{dy}{y} \quad \Longrightarrow \quad \ln|\mu| = \ln|y| + C \quad \Longrightarrow \quad \mu = \pm e^C y$$

gives $\mu(y) = y$.

(c) $\boxed{\text{3 points}}$ Multiply (1) by your $\mu(y)$, then prove that the equation is now exact.

**Solution:** Multiplying (1) by $\mu(y) = y$ gives $y^2 + (2xy - 3y^2e^y)y' = 0$.
Now let $M(x,y) = y^2$ and $N(x,y) = 2xy - 3y^2e^y$. Then we have $M_y = 2y = N_x$. Therefore the equation is now exact.

(d) $\boxed{\text{10 points}}$ Solve (1).

**Solution:** We must find a function $\phi(x,y)$ such that $\phi_x = M = y^2$ and $\phi_y = N = 2xy - 3y^2e^y$. Integrating the former equation gives

$$\phi = \int \phi_x \, dx = \int y^2 \, dx = xy^2 + h(y)$$

for some unknown function $h(y)$. Then differentiating gives

$$2xy - 3y^2e^y = \phi_y = \frac{d}{dy}\left(xy^2 + h(y)\right) = 2xy + h'(y).$$

It follows that $h'(y) = -3y^2e^y$. Using integration by parts, we calculate that

$$h(y) = \int h'(y)\, dy = \int -3y^2e^y \, dy = -3(y^2 - 2y + 2)e^y$$

(where we have chosen the constant of integration as $C = 0$).
Therefore the general solution to (1) is

$$xy^2 - 3(y^2 - 2y + 2)e^y = c$$

for some constant $c$.

2. (a) $\boxed{\text{15 points}}$ **Use the Laplace Transform** to solve $\begin{cases} y'' - 3y' + 2y = \cos t \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$

**Solution:** Taking the Laplace Transform of the ODE gives

$$\begin{aligned}
\mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= \mathcal{L}[\cos t] \\
\left(s^2Y - sy(0) - y'(0)\right) - 3\left(sY - y(0)\right) + 2Y &= \frac{s}{s^2+1} \\
(s^2 - 3s + 2)Y &= \frac{s}{s^2+1} \\
Y &= \frac{s}{(s^2+1)(s^2-3s+2)} \\
&= \frac{s}{(s^2+1)(s-2)(s-1)} \\
&= \frac{As+B}{s^2+1} + \frac{C}{s-2} + \frac{D}{s-1} \\
&= \frac{(As+B)(s-2)(s-1) + C(s^2+1)(s-1) + D(s^2+1)(s-2)}{(s^2+1)(s-2)(s-1)} \\
&\quad \left(A = \tfrac{1}{10},\ B = -\tfrac{3}{10},\ C = \tfrac{2}{5},\ D = -\tfrac{1}{2}\right) \\
&= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} + \frac{\frac{2}{5}}{s-2} - \frac{\frac{1}{2}}{s-1} \\
&= \frac{1}{10}\left(\frac{s}{s^2+1}\right) - \frac{3}{10}\left(\frac{1}{s^2+1}\right) + \frac{2}{5}\left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) \\
&= \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right].
\end{aligned}$$

Therefore the solution to the IVP is

$$y(t) = \frac{1}{10}\cos t - \frac{3}{10}\sin t + \frac{2}{5}e^{2t} - \frac{1}{2}e^{t}.$$

(b) $\boxed{\text{10 points}}$ Find the inverse Laplace Transform of $F(s) = \dfrac{2s-5}{s^2+2s+10}$.

**Solution:**
First we calculate that

$$\begin{aligned}
F(s) = \frac{2s-5}{s^2+2s+10} &= \frac{2s-5}{(s+1)^2+3^2} = \frac{2s+2}{(s+1)^2+3^2} + \frac{-7}{(s+1)^2+3^2} \\
&= 2\left(\frac{s+1}{(s+1)^2+3^2}\right) - \frac{7}{3}\left(\frac{3}{(s+1)^2+3^2}\right) \\
&= 2\mathcal{L}\left[e^{-t}\cos 3t\right] - \frac{7}{3}\mathcal{L}\left[e^{-t}\sin 3t\right].
\end{aligned}$$

Therefore

$$f(t) = \mathcal{L}^{-1}[F](t) = 2e^{-t}\cos 3t - \frac{7}{3}e^{-t}\sin 3t.$$

(a table of Laplace Transforms will be printed here when the answers are hidden)

3. $\boxed{\text{25 points}}$ Solve $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

**Solution:** Clearly the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are $r_1 = r_2 = 1$, since the matrix is triangular.

We calculate that

$$\mathbf{0} = (A - rI)\xi = \begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \xi_2 = 0.$$

Therefore $\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the only linearly independent eigenvector of $A$. $\mathbf{x}^{(1)}(t) = \xi e^{rt} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$ is one solution of the linear system.

For our second solution, we take $\mathbf{x}^{(2)}(t) = \xi t e^t + \eta e^t$ where $\eta$ solves $(A - rI)\eta = \xi$. In other words; where $\eta$ is a generalised eigenvector of $A$.

We calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \xi = (A - rI)\eta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \eta_2 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \eta_2 = 1.$$

Since $\eta_1$ can be any number, so we may choose $\eta_1 = 0$. Then we have $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \xi t e^t + \eta e^t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$.

Therefore the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} t e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \right)$$

for constants $c_1$ and $c_2$.

**Learning Objectives:**

| | | | |
|---|---|---|---|
| LO1 | first order ODEs | 25 points | Q1 |
| LO2 | higher order ODEs | 0 points | |
| LO3 | Laplace T. | 25 points | Q2 |
| LO4 | systems | 50 points | Q3 & Q4 |

4. (a) $\boxed{\text{20 points}}$ Solve $\begin{cases} \mathbf{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$

**Solution:** Solving

$$0 = \det(A - rI) = \begin{vmatrix} 2-r & 1 \\ -1 & 2-r \end{vmatrix} = (2-r)^2 + 1 = 4 - 4r + r^2 + 1 = r^2 - 4r + 5$$

gives

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i.$$

So $\lambda = 2$ and $\mu = 1$.
Next we must find an eigenvector for $r = 2 + i$. We calculate that

$$\mathbf{0} = (A - rI)\xi = \begin{bmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -i\xi_1 + \xi_2 \\ -\xi_1 - i\xi_2 \end{bmatrix}$$

which implies that $0 = -i\xi_1 + \xi_2$. If we choose $\xi_1 = 1$, then we must have $\xi_2 = i\xi_1 = i$. Hence we let $\xi = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Thus

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t}(\cos t + i \sin t) = \begin{bmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{bmatrix} + i \begin{bmatrix} e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}.$$

It follows that the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{2t}$$

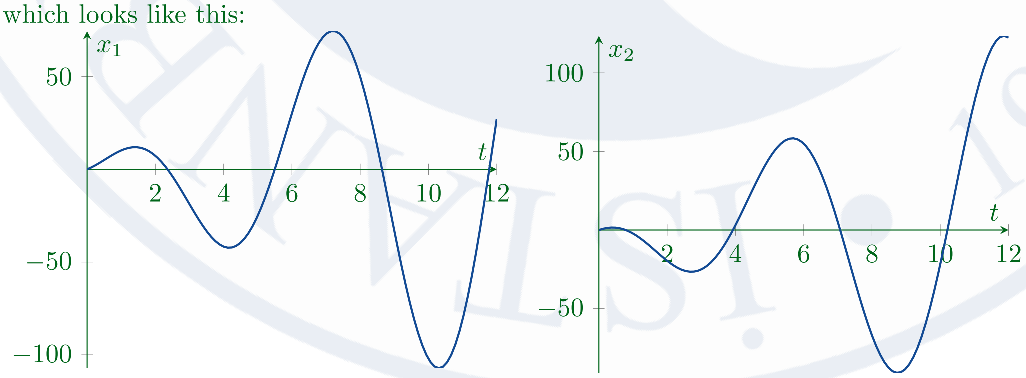
for constants $c_1$ and $c_2$.
Finally we use the initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to find the constants. We calculate that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} \cos 0 \\ -\sin 0 \end{bmatrix} e^0 + c_2 \begin{bmatrix} \sin 0 \\ \cos 0 \end{bmatrix} e^0 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which implies that $c_1 = 1 = c_2$. Therefore the solution is

$$\mathbf{x}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^{2t} + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{2t} = \begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \end{bmatrix} e^{2t}$$

which looks like this:




(b) $\boxed{\text{5 points}}$ Give a fundamental matrix for the above system.

**Solution:** Since the general solution to this linear system is

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) = c_1 \begin{bmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}$$

it follows that a fundamental matrix for this linear system is

$$\Psi(t) = \begin{bmatrix} \mathbf{u}(t) & \mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ -e^{2t}\sin t & e^{2t}\cos t \end{bmatrix}.$$