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STUDENT NO:

DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	28	
2	25	
3	25	
4	22	
Total:	100	

- The time limit is 70 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. Find the Laplace transform $\mathcal{L}\big[f(t)\big](s) = F(s)$ of the following functions.

(a) $\boxed{\text{14 points}}$ $f(t) = \begin{cases} 3, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

Solution: The function f is piecewise continuous and of exponential order for $t > 0$. Since f is defined in two pieces, $\mathcal{L}\{f(t)\}$ is expressed as the sum of two integrals:

$$\begin{aligned} F(s) = \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt = \int_0^{2\pi} e^{-st}(3)\ dt + \int_{2\pi}^\infty e^{-st}(0)\ dt \\ &= \frac{3e^{-st}}{-s}\bigg|_{t=0}^{t=2\pi} + 0 \\ &= \frac{3e^{-2\pi s}}{-s} - \frac{3e^{-s(0)}}{-s} = \boxed{\frac{3}{s}(1 - e^{-2\pi s})} \end{aligned}$$

p.695, pr.37

(b) $\boxed{\text{14 points}}$ $f(t) = te^{2t} + t\cos(3t)$.

Solution: By the linearity of Laplace transform,

$$\begin{aligned} F(s) = \mathcal{L}\{f(t)\} &= \mathcal{L}\{te^{2t} + t\cos(3t)\} = \mathcal{L}\{te^{2t}\} + \mathcal{L}\{t\cos(3t)\} \\ &= -\frac{d}{ds}\mathcal{L}\{e^{2t}\} - \frac{d}{ds}\mathcal{L}\{\cos(3t)\} \\ &= -\frac{d}{ds}\frac{1}{s-2} - \frac{d}{ds}\frac{s}{s^2+9} = (s-2)^{-2} - \frac{s^2+9-2s^2}{(s^2+9)^2} \\ &= \boxed{\frac{1}{(s-2)^2} + \frac{s^2-9}{(s^2+9)^2}} \end{aligned}$$

p.695, pr.37

2. (a) 13 points Use the method of Variation of Parameters to solve $y'' + y = 2\tan t$, $-\pi/2 < t < \pi/2$,

Solution: The characteristic equation is

$$m^2 + 1 = 0$$

so we have conjugate roots $r = \pm i$. The general homogeneous solution is then

$$y = c_1 \cos t + c_2 \sin t$$

with two arbitrary constants c_1 and c_2. Using the method of variation of parameters, we look for a particular solution y_p in the form

$$y_p = v_1 \cos t + v_2 \sin t$$

The two linearly independent solutions are $y_1 = \cos t$ and $y_2 = \sin t$ and their Wronskian is

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

We find v_1' and v_2' as (the Cramer rule)

$$v_1' = \begin{vmatrix} 0 & \sin t \\ 2\tan t & \cos t \end{vmatrix} = -2\tan t \sin t = \frac{2\cos^2 t - 2}{\cos t} = 2\cos t - 2\sec t \text{ and } v_2' = \begin{vmatrix} \cos t & 0 \\ -\sin t & 2\tan t \end{vmatrix} = 2\sin t.$$

Integrating the last expressions, we get

$$v_1 = \int (2\cos t - 2\sec t)\, dt = 2\sin t - 2\ln|\sec t + \tan t| \text{ and } v_2 = \int (2\sin t)\, dt = -2\cos t$$

Thus the general solution is

$$\boxed{y = c_1 \cos t + c_2 \sin t + \cos t(2\sin t - 2\ln|\sec t + \tan t|) + 2\sin t\cos t, \text{ for } -\pi/2 < t < \pi/2}.$$

p.491, pr.86

(b) 12 points Use the method of Undetermined Coefficients to solve $y''' - y' = 4t$.

Solution: The characteristic equations

$$0 = r^3 - r = r(r^2 - 1) = r(r-1)(r+2)$$

has roots $r_1 = 0$, $r_2 = -1$ and $r_3 = 1$. Therefore the general solution to the homogeneous equation $y''' - y' = 0$ is

$$y(t) = c_1 + c_2 e^{-t} + c_3 e^t.$$

Next we must find a particular solution to the non-homogeneous equation. Since $4t$ is a polynomial, we try the ansatz $Y(t) = At^2 + Bt + C$. Since $Y' = 2At + B$ and $Y'' = 0$, we get

$$4t = Y''' - Y = 0 - 2At - B.$$

Clearly we require $A = -2$ and $B = 0$. C could take any value, so we choose $C = 0$ for simplicity. Therefore

$$Y(t) = -2t^2$$

is a particular solution.
The general solution to the non-homogeneous equation is thus

$$y(t) = c_1 + c_2 e^{-t} + c_3 e^t - 2t^2.$$

p.695, pr.37

3. (a) $\boxed{\text{15 points}}$ Find all solutions of the equation $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$.

Solution: This is a homogeneous equation which can be written as

$$\frac{dy}{dx} = \frac{(y/x)-4}{1-(y/x)}.$$

The change of variables $v = y/x$, $y = xv$, $dy = xdv + vdx$ leads to

$$x\frac{dv}{dx} + v = \frac{v-4}{1-v}$$

which is a separable equation. The general solution:

$$\begin{aligned} x\frac{dv}{dx} = \frac{v-4}{1-v} - v &= \frac{v^2-4}{1-v}; \\ \frac{1-v}{v^2-4}dv &= \frac{dx}{x}; \\ \int \frac{1-v}{v^2-4}dv &= \int \frac{dx}{x} \end{aligned}$$

Partial fractions gives

$$\begin{aligned} \frac{-v+1}{v^2-4} &= \frac{A}{(v-2)} + \frac{B}{v+2}; \\ -v+1 = A(v+2) + B(v-2) \Rightarrow A &= -1/4, B = -3/4; \\ \int \frac{-1/4}{v-2}dv + \int \frac{-3/4}{v+2}dv &= \int \frac{dx}{x}; \\ \ln(|v-2|^{-1/4}|v+2|^{-3/4}) &= \ln|x| + C; \\ \frac{1}{(v-2)(v+2)^3} &= Cx^4; \\ x^4(v-2)(v+2)^3 &= C. \end{aligned}$$

Note that the last formula includes the singular solutions $v = 2$ and $v = -2$. Substituting $y = vx$ gives

$$\boxed{(y-2x)(y+2x)^3 = C}$$

p.583, pr.17

(b) $\boxed{\text{10 points}}$ $ty' + y = 2\sin t$, $y(\pi/2) = 1$, $t > 0$.

Solution:
First we write the ODE in the standard form

$$y' + \left(\frac{1}{t}\right)y = \frac{2\sin t}{t}.$$

We require the integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t.$$

Multiplying our ODE by t puts us back at the original ODE:

$$ty' + y = 2\sin t.$$

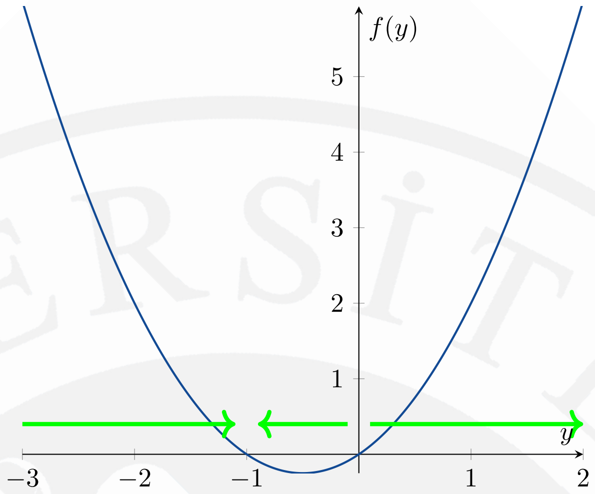
Then we calculate that

$$\begin{aligned} ty' + y &= 2\sin t \\ (ty)' &= 2\sin t \\ ty &= 2\int \sin t\,dt = -2\cos t + C \\ y &= -\frac{2}{t}\cos t + \frac{C}{t}. \end{aligned}$$

p.583, pr.17

4. (a) $\boxed{\text{12 points}}$ Find all the equilibrium solutions/critical points of $y' = y(1+y)$. Determine whether each one is asymptotically stable, unstable or semistable.

Solution:



The graph of $f(y) = y(1+y)$ is shown above. Clearly

$$0 = y' = y(y+1) \implies y = 0 \text{ or } 1.$$

We can see from the graph that $y = -1$ is an asymptotically stable equilibrium solution, while $y = 0$ is an unstable equilibrium solution.

p.573, pr.38

(b) $\boxed{\text{10 points}}$ Draw a direction field for $\dfrac{dy}{dx} = y(1+y)$.

Solution: