Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 18 | |
| 3 | 32 | |
| 4 | 25 | |
| Total: | 100 | |

1. 25 Points Solve the Initial Value Problem

$$y'' + 4y = f(t) = \begin{cases} t, & 0 \le t < 3, \\ 1, & 3 \le t < \infty; \end{cases} \quad y(0) = 0, \;\; y'(0) = 0$$

**Solution:** The function $f$ can be written as $f(t) = t - t\mathscr{U}(t-3) + \mathscr{U}(t-3)$, so by linearity, we have

$$\mathscr{L}\{y''\} + 4\mathscr{L}\{y\} = \mathscr{L}\{t - (t-3)\mathscr{U}(t-3) - 2\mathscr{U}(t-3)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-3s} - \frac{3}{s}e^{-3s} + \frac{1}{s}e^{-3s}$$

$$(s^2+4)Y(s) = \frac{1}{s^2} - \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-3s}$$

$$Y(s) = \frac{1}{s^2(s^2+4)} - \left(\frac{2s+1}{s^2(s^2+4)}\right)e^{-3s}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} - \mathscr{L}^{-1}\left\{\left(\frac{2s+1}{s^2(s^2+4)}\right)e^{-3s}\right\}$$

$$y(t) = \boxed{\frac{1}{4}t - \frac{1}{8}\sin(2t) - \left(\frac{1}{4} - \frac{1}{4}(t-3) - \frac{1}{4}\cos 2(t-3) + \frac{1}{8}\sin 2(t-3)\right)\mathscr{U}(t-3)}$$

where we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

and

$$\mathscr{L}^{-1}\left\{\frac{2s+1}{s^2(s^2+4)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{s}{s^2+4} + \frac{1}{8}\frac{2}{s^2+4}\right\} = \frac{1}{4} - \frac{1}{4}t - \frac{1}{4}\cos(2t) + \frac{1}{8}\sin(2t)$$

p.695, pr.37

2. $\boxed{\text{18 Points}}$ Find the inverse Laplace transform of

$$F(s) = \frac{3s+3}{s^2+2s+5}.$$

**Solution:** Notice that the denominator $s^2+2s+5$ is irreducible over the reals. Completing the square, $s^2+2s+5 = (s+1)^2+4$. Now convert the function into a rational function of the variable $u = s+1$. That is,

$$\frac{3s+3}{s^2+2s+5} = \frac{3(s+1)}{(s+1)^2+4}$$

We know that

$$\mathscr{L}^{-1}\left\{\frac{3u}{u^2+4}\right\} = 3\cos(2t).$$

Using the fact that $\mathscr{L}\left\{e^{at}f(t)\right\} = [\mathscr{L}\left\{f(t)\right\}]_{s\to s-a}$,

$$\mathscr{L}^{-1}\left\{\frac{3s+3}{s^2+2s+5}\right\} = 3e^{-t}\cos(2t).$$

p.491, pr.86

| $f(t)$ | $F(s) = \mathscr{L}[f](s)$ | |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $s>0$ |
| $e^{at}$ | $\frac{1}{s-a}$ | $s>a$ |
| $t^n \quad (n\in\mathbb{N})$ | $\frac{n!}{s^{n+1}}$ | $s>0$ |
| $\sin at$ | $\frac{a}{s^2+a^2}$ | $s>0$ |
| $\cos at$ | $\frac{s}{s^2+a^2}$ | $s>0$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}$ | $s>\lvert a\rvert$ |
| $\cosh at$ | $\frac{s}{s^2-a^2}$ | $s>\lvert a\rvert$ |
| $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$ | $s>a$ |
| $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$ | $s>a$ |
| $t^n e^{at} \quad (n\in\mathbb{N})$ | $\frac{n!}{(s-a)^{n+1}}$ | $s>a$ |
| $u_c(t)$ | $\frac{e^{-cs}}{s}$ | $s>0$ |
| $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ | |
| $e^{ct}f(t)$ | $F(s-c)$ | |
| $f(ct) \quad (c>0)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$ | |
| $\int_0^t f(t-\tau)g(\tau)d\tau$ | $F(s)G(s)$ | |
| $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ | |

3. Find the *inverse Laplace transform* of

$$F(s) = \frac{1}{(s+1)(s-2)}.$$

(a) 16 Points Using partial fractions.

**Solution:** We write

$$\begin{aligned}
F(s) &= \frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} \\
1 &= A(s-2) + B(s+1) \\
1 &= As - 2A + Bs + B \Rightarrow 1 = (A+B)s - 2A + B \\
&\Rightarrow A + B = 0, \ \ -2A + B = 1 \Rightarrow B = 1/3, A = -1/3 \\
&= \frac{1}{(s+1)(s-2)} = \frac{-1/3}{s+1} + \frac{1/3}{s-2} \\
&\Rightarrow \mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \mathscr{L}^{-1}\left\{\frac{-1/3}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{1/3}{s-2}\right\} = \boxed{\frac{1}{3}\left(e^{2t} - e^{-t}\right)}
\end{aligned}$$

p.583, pr.17

(b) 16 Points Using Convolution theorem.

**Solution:** Let $G(s) = \frac{1}{s+1}$ and $H(s) = \frac{1}{s-2}$ so that

$$g(t) = e^{-t} \text{ and } h(t) = e^{2t}.$$

By the Convolution theorem,

$$\begin{aligned}
f(t) &= \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\{G(s)H(s)\} \\
&= g * h = e^{-t} * e^{2t} \\
&= \int_0^t e^{-\tau} e^{2(t-\tau)}\, d\tau = \int_0^t e^{2t-3\tau} \cos\tau \, d\tau = -\frac{1}{3}\left[e^{2t-3\tau}\right]_0^t = \boxed{\frac{1}{3}\left(e^{2t} - e^{-t}\right)}
\end{aligned}$$

p.583, pr.17

4. $\boxed{\text{25 Points}}$ Find the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}.$$

**Solution:** We first find the eigenvalues and eigenvectors of the matrix of coefficients.
From the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-2-\lambda) - (1)(1) = 4 + 4\lambda + \lambda^2 - 1 = \lambda^2 4\lambda + 3 = (\lambda+3)(\lambda+1) = 0$$

we see that the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$.

Now for $\lambda_1 = -3$, $(A - \lambda_1 I)\mathbf{K} = 0$ is equivalent to

$$\begin{aligned} k_1 + k_2 &= 0 \\ k_1 + k_2 &= 0. \end{aligned}$$

Thus $k_1 = -k_2$. When $k_2 = 1$ the related eigenvector is

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = -1$, we have

$$\begin{aligned} -k_1 + k_2 &= 0 \\ k_1 - k_2 &= 0. \end{aligned}$$

so $k_1 = k_2$ therefore with $k_2 = 1$ the corresponding eigenvector

$$\mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since the matrix of coefficients $A$ is a $2 \times 2$ matrix and since we have found two linearly independent solutions,

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} \text{ and } \mathbf{X}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t},$$

we conclude that the general solution of the system is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t},$$

p.573, pr.38