2016–17 **MAT461 Fonksiyonel Analiz I – Ödev 3** N. Course

SON TESLİM TARİHİ: Çarşamba 30 Kasım 2016 saat 10:00'e kadar.

Egzersiz 5 (Norms).

(a) [20p] Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $f, g \in X$ and $g \neq 0$. Show that
$$\|f+g\| = \|f\| + \|g\| \qquad \Longleftrightarrow \qquad f = \alpha g \quad \text{for some } \alpha \in \mathbb{R},\ \alpha \geq 0.$$

(b) [20p] Show that the maximum norm, $\|\cdot\|_\infty$, on $C([0,1];\mathbb{R})$ does not satisfy the parallelogram law.

Egzersiz 6 (Integral Operators). Let $k : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function. We can define an operator, K, by
$$\big(Kf\big)(x) = \int_0^1 k(x,y) f(y)\, dy$$
for all $f \in C([0,1])$.

(a) [15p] Show that $K : \big(C([0,1]), \|\cdot\|_\infty\big) \to \big(C([0,1]), \|\cdot\|_\infty\big)$ is a bounded operator.

(b) [15p] Show that $K : \big(C([0,1]), \|\cdot\|_{L^2}\big) \to \big(C([0,1]), \|\cdot\|_{L^2}\big)$ is a bounded operator.
[HINT: If g, h are continuous, then $|\int \bar{g}h| \leq (\int |g|^2)^{\frac{1}{2}} (\int |h|^2)^{\frac{1}{2}}$ by Cauchy-Schwarz. Setting $g \equiv 1$ then gives ? $\leq$?]

Egzersiz 7 (Multiplication of Operators). Let $(X, \|\cdot\|_X)$ be a normed vector space, let
$$\mathcal{B}(X) := \{\text{all bounded linear operators } X \to X\}$$
and let
$$\|A\| := \sup_{\substack{f \in \mathcal{D}(A) \\ \|f\|_X = 1}} \|Af\|_X$$
be the operator norm.

(a) [15p] Show that $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathcal{B}(X)$.

(b) [15p] Show that multiplication is continuous: In other words; show that if $A_n, A, B_n, B \in \mathcal{B}(X)$, $A_n \to A$ and $B_n \to B$ then $A_n B_n \to AB$.

3. $\|f\| = \|f-g+g\| \leq \|f-g\| + \|g\|$ so $\|f\| - \|g\| \leq \|f-g\|$. Similarly $\|g\| - \|f\| \leq \|g-f\| = \|f-g\|$. Therefore $|\|f\| - \|g\|| \leq \|f-g\|$

4. (a) First we must show that $\ell^\infty(\mathbb{N})$ is a vector space: If $a = (a_j)_{j=1}^\infty$ and $b = (b_j)_{j=1}^\infty$ are elements of $\ell^\infty(\mathbb{N})$, and if $\lambda \in \mathbb{C}$, then $\|a+\lambda b\|_\infty = \sup_j |a_j + \lambda b_j| \leq \sup_j |a_j| + |\lambda| \sup_j |b_j| = \|a\|_\infty + |\lambda| \|b\|_\infty < \infty$, so $a + \lambda b \in \ell^\infty(\mathbb{N})$.

   (b) Clearly $\|a\|_\infty > 0$ for all $a \in \ell^\infty(\mathbb{N})$, $a \neq 0$ (i.e. not all $a_n = 0$). The triangle inequality was shown in part (a). Finally $\|\lambda a\|_\infty = \sup_j |\lambda a_j| = |\lambda| \sup_j |a_j| = |\lambda| \|a\|_\infty$ for all $a \in \ell^\infty(\mathbb{N})$ and for all $\lambda \in \mathbb{C}$. Therefore $\|\cdot\|_\infty$ is a norm on $\ell^\infty(\mathbb{N})$.

   (c) We must show that $\ell^\infty(\mathbb{N})$ is complete: Let $a^n = (a_j^n)_{j=1}^\infty$ be a Cauchy sequence in $\ell^\infty(\mathbb{N})$ (i.e. a sequence of sequences). Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $m, n > N \implies \|a^m - a^n\|_\infty < \varepsilon \implies \left|a_j^m - a_j^n\right| < \varepsilon$ for all j. For each j, $(a_j^n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, there exists a limit $a_j = \lim_{n\to\infty} a_j^n$. Since $\left|a_j^m - a_j^n\right| < \varepsilon$ for all $m, n > N$ and all j, we must also have that $\left|a - a_j^n\right| \leq \varepsilon$ for all $n > N$ and j. It follows that $\|a - a^n\|_\infty = \sup_j \left|a_j - a_j^n\right| \leq \varepsilon$ for all $n > N$ and so $\|a\|_\infty \leq \|a - a^n\|_\infty + \|a^n\|_\infty < \infty$. Therefore $a \in \ell^\infty(\mathbb{N})$ and so $\ell^\infty(\mathbb{N})$ is complete.

   (d) Let $Q = \{a = (a_j)_{j=1}^\infty \in \ell^\infty(\mathbb{N}) : a_j \in \{0,1\}\ \forall j\} \subseteq \ell^\infty(\mathbb{N})$. If $a, b \in Q$ and $a \neq b$ then we must have that $\|a - b\|_\infty = 1$. Suppose that Q is countable. Then we can label every element of Q as $a^1, a^2, a^3, a^4, \ldots$. Now define a new sequence b as follows: if $a_j^j = 1$ then define $b_j = 0$, otherwise define $b_j = 1$. Then for every n, $b \neq a^n$ (because $b_n \neq a_n^n$) so b is not in our list $a^1, a^2.a^3, a^4, \ldots$. However, every term in the sequence of b is either 1 or 0 so b must be in Q. Contradiction. Therefore Q is uncountable.
   Finally suppose that $P = \{p^1, p^2, p^3, \ldots\}$ is a countable, dense subset of $\ell^\infty(\mathbb{N})$ and let $0 < \varepsilon < \frac{1}{10}$. Then for every $a \in Q$ there must exists some $p \in P$ such that $\|a - p\|_\infty < \varepsilon$. But since $a, b \in Q \implies \|a - b\|_\infty = 1$, each $p \in P$ can only be close to at most one element in Q. But Q is uncountable so P must be uncountable as well. Contradiction. Therefore $\ell^\infty(\mathbb{N})$ is not separable.