OKAN ÜNİVERSİTESİ
MÜHENDİSLİK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ



SON TESLİM TARİHİ: Çarşamba 21 Aralık 2016 saat 10:00'e kadar.

**Egzersiz 8 (Unitary Operators).** [25p] Let $X$ be a Hilbert space. Suppose that $U : X \to X$ is unitary and $M \subseteq X$. Show that $U(M^\perp) = (UM)^\perp$.

**Egzersiz 9 (Orthogonal Projection).** [15p] Let $X$ be a Hilbert space and $M \subseteq X$ be a subspace. Let $\Pi_M : X \to M$ be orthogonal projection. Suppose that $\Pi_M \neq 0$ (in other words, suppose that $\exists f \in X$ such that $\Pi_M f \neq 0$). Show that $\|\Pi_M\| = 1$.

**Egzersiz 10 (Uniqueness of Orthogonal Projection).** [5×6p] Let $X$ be a Hilbert space. Suppose that $P \in \mathcal{B}(X)$ satisfies

$$P^2 = P \qquad \text{and} \qquad \langle Pf, g\rangle = \langle f, Pg\rangle \ \forall f, g \in X.$$

Let $M := \operatorname{Ran}(P)$ and let $\Pi_M : X \to M$ denote orthogonal projection.
Show that

(a) $Pf = f$ for all $f \in M$;

(b) $M$ is closed;

(c) $g \in M^\perp \implies Pg \in M^\perp$;

(d) $g \in M^\perp \implies Pg = 0$; and

(e) $P = \Pi_M$.

**Egzersiz 11 (Adjoints).** Let $X$ be a Hilbert space and let $u, v \in X$. Let $A : X \to X$ be an operator defined by

$$Af := \langle u, f\rangle\, v.$$

(a) [10p] Show that $A$ is bounded.

(b) [10p] Calculate $\|A\|$.

(c) [10p] Calculate the adjoint of $A$.

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*Ödev 3'ün çözümleri*

5. (a) Given that $\|f+g\|^2 = \langle f+g, f+g\rangle = \|f\|^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|^2$ and $(\|f\|+\|g\|)^2 = \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2$, we can see that $\|f+g\| = \|f\| + \|g\|$ if and only if $2\operatorname{Re}\langle f,g\rangle = \langle f,g\rangle + \langle g,f\rangle = 2\|f\|\,\|g\|$.

   But $\operatorname{Re}\langle f,g\rangle \le |\langle f,g\rangle| \le \|f\|\,\|g\|$ by the Cauchy-Schwarz Inequality. The second "$\le$" is an "$=$" if and only if $f = \alpha g$ for some $\alpha \in \mathbb{C}$ (by the Cauchy-Schwarz Inequality). Since $\langle \alpha g, g\rangle = \bar{\alpha}\|g\|^2$, the first "$\le$" is an "$=$" if and only if $\alpha \in \mathbb{R}$ and $\alpha \ge 0$.

   (b) Define $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x$ and $g(x) = 2x - 1$. Then $\|f+g\|_\infty = |f(1)+g(1)| = 2$ and $\|f-g\|_\infty = |f(0)-g(0)| = 1$, so $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 9$. However $\|f\|_\infty = 1$ and $\|g\|_\infty = 1$, so $2\|f\|_\infty^2 + 2\|g\|_\infty^2 = 8 \neq \|f+g\|_\infty^2 + \|f-g\|_\infty^2$. Therefore the maximum norm $\|\cdot\|_\infty$ does not satisfy the parallelogram law.

6. (a) First, $k : [0,1]\times[0,1] \to \mathbb{C}$ is continuous, so $\|k\|_\infty < \infty$. Let $f \in C([0,1])$. Then $|(Kf)(x)| = \left|\int_0^1 k(x,y)f(y)\,dy\right| \le \int_0^1 |k(x,y)|\,|f(y)|\,dy \le \|k\|_\infty \|f\|_\infty$. Therefore $\|K\| = \sup_{f\in\mathcal{D}(K),\ \|f\|_\infty=1} \|(Kf)(x)\|_\infty \le \|k\|_\infty < \infty$. So $K$ is a bounded operator.

   (b) Let $f \in C([0,1])$. Then $\|(Kf)(x)\|_{L^2}^2 = \int_0^1 |(Kf)(x)|^2\,dx = \int_0^1 \left|\int_0^1 k(x,y)f(y)\,dy\right|^2 dx$
   $\le \int_0^1 \left(\int_0^1 |k(x,y)|\,|f(y)|\,dy\right)^2 dx \le \int_0^1\int_0^1 |k(x,y)|^2 |f(y)|^2\,dy\,dx \le \|k\|_\infty^2 \int_0^1\int_0^1 |f(y)|^2\,dy\,dx = \|k\|_\infty^2 \int_0^1 \|f(y)\|_{L^2}^2\,dx$
   $= \|k\|_\infty \|f(y)\|_{L^2}^2$. Therefore $\|K\| \le \|k\|_\infty < \infty$ as above. So $K$ is a bounded operator.

7. (a) By definition of the operator norm, $\|Bf\|_X \le \|B\|\,\|f\|_X$ for all $f \in X$. But $Bf \in X$, so $\|ABf\|_X \le \|A\|\,\|Bf\|_X \le \|A\|\,\|B\|\,\|f\|_X$ for all $f \in X$. It follows that $\|AB\| = \sup_{f\in\mathcal{D}(AB),\ \|f\|_X=1} \|ABf\|_X$
   $\le \sup_{f\in\mathcal{D}(AB),\ \|f\|_X=1} \|A\|\,\|B\|\,\|f\|_X \le \|A\|\,\|B\|$.

   (b) Let $\varepsilon > 0$. Suppose $A_n \to A$ and $B_n \to B$. Then $M := \max_n\{\|B_n\|, \|B\|\} < \infty$. There exists $N$ such that $\|A_n - A\| < \frac{\varepsilon}{2M}$ and $\|B_n - B\| < \frac{\varepsilon}{2\|A\|}$ for all $n > N$. But then $\|A_nB_nf - ABf\|_X \le \|A_nB_nf - AB_nf\|_X + \|AB_nf - ABf\|_X \le \|A_n - A\|\,\|B_nf\|_X + \|A\|\,\|B_nf - Bf\|_X \le \|A_n - A\|\,\|B_n\|\,\|f\|_X + \|A\|\,\|B_n - B\|\,\|f\|_X < \frac{\varepsilon}{2M} M \|f\|_X + \|A\| \frac{\varepsilon}{2\|A\|}\|f\|_X = \varepsilon\|f\|_X$ for all $n > N$. Therefore $n > N \implies \|A_nB_n - AB\| < \varepsilon$. So $A_nB_n \to AB$.