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MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

03.01.2013 **MAT 461 – Fonksiyonel Analiz I – Final Sınavın Çözümleri** N. Course

Question 1. Let X be a vector space

(a) [4p] Give the definition of a *norm* on X.

A *norm* is a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies

(i) $\|f\| > 0$ for all $f \in X$, $f \neq 0$;

(ii) $\|\alpha f\| = |\alpha| \, \|f\|$ for all $f \in X$, $\alpha \in \mathbb{C}$ (or $\alpha \in \mathbb{R}$ if X is a real vector space);

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$.

Consider the set $\ell^1(\mathbb{N}) := \{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_1 < \infty\}$ where $\|a\|_1 := \sum_{j=1}^\infty |a_j|$.

(b) [5p] Show that $\ell^1(\mathbb{N})$ is a vector space

Let $a, b \in \ell^1(\mathbb{N})$ and $\lambda \in \mathbb{C}$. Then $\|a\|_1 < \infty$ and $\|b\|_1 < \infty$. So

$$\|a + \lambda b\|_1 = \sum_j |a_j + \lambda b_j| \leq \sum_j |a_j| + |\lambda| \, |b_j| = \sum_j |a_j| + |\lambda| \sum_j |b_j| = \|a\|_1 + |\lambda| \, \|b\|_1 < \infty.$$

Therefore $a + \lambda b \in \ell^1(\mathbb{N})$. Hence $\ell^1(\mathbb{N})$ is a vector space.

(c) [5p] Show that $\|\cdot\|_1$ is a norm on $\ell^1(\mathbb{N})$.

I proved the triangle inequality in part (b). (i) Suppose $a \in \ell^1(\mathbb{N})$ and $a \neq 0$. Then $\exists j$ such that $|a_j| > 0$. So $\|a\|_1 \geq |a_j| > 0$. (ii) Finally suppose that $\alpha \in \mathbb{C}$ and $a \in \ell^1(\mathbb{N})$. Then $\|\alpha a\|_1 = \sum_j |\alpha a_j| = |\alpha| \sum_j |a_j| = |\alpha| \, \|a\|_1$.
Therefore $\|\cdot\|_1$ is a norm on $\ell^1(\mathbb{N})$.

(d) [2p] Give the definition of a *Banach space*.

A *Banach space* is a complete normed vector space.

(e) [9p] Show that $(\ell^1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space.

I have already proved that $\ell^1(\mathbb{N})$ is a vector space and that $\|\cdot\|_1$ is a norm on $\ell^1(\mathbb{N})$. It remains to prove that $\ell^1(\mathbb{N})$ is complete with this norm.

Let $x^n = (x_j^n)_{j=1}^\infty$ be a Cauchy sequence in $\ell^1(\mathbb{N})$. So for any $\varepsilon > 0$ we can find an $N = N(\varepsilon)$ such that $\|x^m - x^n\|_1 < \varepsilon$ for $m, n > N$. This implies that $\left|x_j^m - x_j^n\right| < \varepsilon$ for all j. Thus, for each fixed j, the sequence $(x_j^n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, this sequence has a limit: Define $x_j := \lim_{n\to\infty} x_j^n$.
Now consider $\sum_{j=1}^k \left|x_j^m - x_j^n\right| < \varepsilon$. Letting $m \to \infty$ gives us $\sum_{j=1}^k \left|x_j - x_j^n\right| \leq \varepsilon$. This is true for all k. So letting $k \to \infty$ we have $\|x - x^n\|_1 \leq \varepsilon$. Hence $(x - x^n) \in \ell^1(\mathbb{N})$. Since $x^n \in \ell^1(\mathbb{N})$, it follows that $x = x^n + (x - x^n) \in \ell^1(\mathbb{N})$ also.
Therefore, every Cauchy sequence in $\ell^1(\mathbb{N})$ is convergent. Hence $\ell^1(\mathbb{N})$ is complete.

Question 2 (Orthonormal sets)**.** Let X be a Hilbert space.

(a) [5p] Give the definition of an *orthonormal set.*

A set $\{u_j\}$ is called *orthonormal* iff

$$\langle u_j, u_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Let $\{u_j\}_{j=1}^n \subseteq X$ be an orthonormal set and let $f \in X$. Define $f_\parallel = \sum_{j=1}^n \langle u_j, f \rangle u_j$ and $f_\perp = f - f_\parallel$.

(b) [5p] Show that

$$\langle u_j, f_\perp \rangle = 0$$

for all $j = 1, 2, \ldots, n$.

$$\begin{aligned} \langle u_j, f_\perp \rangle = \langle u_j, f \rangle - \langle u_j, f_\parallel \rangle &= \langle u_j, f \rangle - \left\langle u_j, \sum_{k=1}^n \langle u_k, f \rangle u_k \right\rangle \\ &= \langle u_j, f \rangle - \sum_{k=1}^n \langle u_k, f \rangle \langle u_j, u_k \rangle = \langle u_j, f \rangle - \langle u_j, f \rangle = 0 \end{aligned}$$

since $\langle u_j, u_k \rangle = 0$ if $j \neq k$ and $= 1$ if $j = k$.

(c) [5p] Use (b) to show that $f_\parallel$ and $f_\perp$ are orthogonal.

$$\langle f_\parallel, f_\perp \rangle = \left\langle \sum_{j=1}^n \langle u_j, f \rangle u_j, f_\perp \right\rangle = \sum_{j=1}^n \langle u_j, f \rangle \langle u_j, f_\perp \rangle = 0$$

by part (b).

(d) [5p] Show that

$$\|f\|^2 = \left\|f_\parallel + f_\perp\right\|^2 = \sum_{j=1}^n |\langle u_j, f \rangle|^2 + \|f_\perp\|^2 .$$

[HINT: Do you remember Pythogora's Theorem?]

Since $f_\parallel$ and $f_\perp$ are orthogonal, we know by Pythogora that $\left\|f_\parallel + f_\perp\right\|^2 = \left\|f_\parallel\right\|^2 + \|f_\perp\|^2$. Moreover

$$\begin{aligned} \left\|f_\parallel\right\|^2 = \langle f_\parallel, f_\parallel \rangle &= \left\langle \sum_{j=1}^n \langle u_j, f \rangle u_j, \sum_{k=1}^n \langle u_k, f \rangle u_k \right\rangle \\ &= \sum_{j=1}^n \sum_{k=1}^n \langle \langle u_j, f \rangle u_j, \langle u_k, f \rangle u_k \rangle \\ &= \sum_{j=1}^n \sum_{k=1}^n \overline{\langle u_j, f \rangle} \langle u_k, f \rangle \langle u_j, u_k \rangle = \sum_{j=1}^n |\langle u_j, f \rangle|^2 . \end{aligned}$$

Now suppose that $g \in \operatorname{span}\{u_j\}_{j=1}^n$. [HINT: This means that $g = g_\parallel$ and $g_\perp = 0$.]

(e) [5p] Show that

$$\|f - g\| \geq \|f_\perp\| .$$

Since $(f - g) = (f - g)_\parallel + (f - g)_\perp = (f_\parallel - g_\parallel) + (f_\perp - g_\perp) = (f_\parallel - g_\parallel) + (f_\perp - 0)$, it follows that

$$\|f - g\|^2 = \left\|f_\parallel - g_\parallel\right\|^2 + \|f_\perp\|^2 \geq \|f_\perp\|^2 .$$

Question 3 (The Operator Norm)**.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let $A : X \to Y$ be an operator.

(a) [3p] Give the definition of the *operator norm* of A

$$\|A\| = \sup_{f \in X,\ \|f\|_X = 1} \|Af\|_Y .$$

(b) [5p] Let $\lambda > 0$. Suppose that $\|Af\|_Y \leq \lambda \|f\|_X$ for all $f \in X$. Show that $\|A\| \leq \lambda$.

$$\|A\| = \sup_{\|f\|_X = 1} \|Af\|_Y \leq \sup_{\|f\|_X = 1} \lambda \|f\|_X = \lambda \sup_{\|f\|_X = 1} \|f\|_X = \lambda.$$

(c) [5p] Show that $\|Af\|_Y \leq \|A\| \|f\|_X$ for all $f \in X$.

The result is clearly true if $f = 0$. Suppose $f \neq 0$. Then

$$\begin{aligned} \|Af\|_Y = \frac{\|f\|_X}{\|f\|_X} \|Af\|_Y = \|f\|_X \left\| \frac{1}{\|f\|_X} Af \right\|_Y = \|f\|_X \left\| A \left(\frac{f}{\|f\|_X} \right) \right\|_Y \\ \leq \|f\|_X \sup_{\|g\|_X = 1} \|Ag\|_Y = \|f\|_X \|A\| . \end{aligned}$$

(d) [2p] Give the definition of a *bounded operator*.

A is called a *bounded operator* if $\|A\| < \infty$.

Now suppose that $A : X \to X$ and $B : X \to X$ are bounded operators.

(e) [10p] Show that $\|AB\| \leq \|A\| \|B\|$.

It follows by part (c) that

$$\|ABf\|_X \leq \|A\| \|Bf\|_X \leq \|A\| \|B\| \|f\|_X$$

for all f. Therefore $\|AB\| \leq \|A\| \|B\|$ by part (b).

Question 4 (Orthonormal Bases)**.** Let X be a Hilbert space. Let $\{u_j\}_{j\in J}$ be an orthonormal **set** in X.

(a) [5p] Suppose that

$$\|f\|^2 = \sum_{j\in J} |\langle u_j, f\rangle|^2$$

for all $f \in X$. Show that

$$\langle u_j, f\rangle = 0 \quad \forall j \in J \qquad \Longrightarrow \qquad f = 0.$$

Suppose that $\langle u_j, f\rangle = 0 \ \forall \ j \in J$. Then

$$\|f\|^2 = \sum_{j\in J} |\langle u_j, f\rangle|^2 = \sum_{j\in J} 0 = 0$$

which implies that $f = 0$.

(b) [7p] Now suppose that

$$\langle u_j, f\rangle = 0 \quad \forall j \in J \qquad \Longrightarrow \qquad f = 0$$

is true. Suppose that $Q \supseteq \{u_j\}_{j\in J}$ is an orthonormal set. Show that $Q = \{u_j\}_{j\in J}$.
[HINT: Use proof by contradiction. Start by supposing that $\exists\ g \in Q$ such that $g \notin \{u_j\}_{j\in J}$.]

Suppose that $Q \neq \{u_j\}_{j\in J}$. Then there exists $g \in Q \setminus \{u_j\}_{j\in J}$ where $g \neq 0$. However, since Q is orthonormal, we know that $\langle u_j, g\rangle = 0$ for all $j \in J$. By the assumption in the question, this implies that $g = 0$. Contradiction.

Now suppose that:

- $\{u_j\}_{j\in J}$ is an orthonormal **basis** of X;
- $A, B \in \mathcal{B}(X)$
- we define $A_{jk} := \langle u_j, Au_k\rangle$ and $B_{jk} := \langle u_j, Bu_k\rangle$, for all $j, k \in J$.

(c) [13p] If $A_{jk} = B_{jk}$ for all $j, k \in J$, show that $A = B$.
[HINT: By a theorem from the course: $\langle u_j, f\rangle = 0\ \forall j \implies f = 0$. Use this to show that $(A - B)g = 0\ \forall g \in X$.]

Let $g \in X$. Since $\{u_j\}_{j\in J}$ is an orthonormal basis, we know that

$$g = \sum_k \langle u_k, g\rangle\, u_k.$$

Now, for any $j \in J$, we have that

$$\begin{aligned}\langle u_j, (A-B)g\rangle = \langle u_j, Ag\rangle - \langle u_j, Bg\rangle &= \sum_k \langle u_k, g\rangle \langle u_j, Au_k\rangle - \sum_k \langle u_k, g\rangle \langle u_j, Bu_k\rangle \\ &= \sum_k \langle u_k, g\rangle\, A_{jk} - \sum_k \langle u_k, g\rangle\, B_{jk} = \sum_k \langle u_k, g\rangle\, (A_{jk} - B_{jk}) = 0.\end{aligned}$$

Then by the hint, it follows that $(A - B)g = 0$. Since this is true for all g, we must have that $A = B$.

Question 5 (Adjoints of Operators)**.** Let X be a Hilbert space and let $\mathcal{B}(X) = \{A : X \to X : A$ is linear and bounded$\}$.

(a) [5p] Let $A \in \mathcal{B}(X)$ be an operator. Give the definition of the *adjoint* of A.

The *adjoint* of A is the unique operator $A^* \in \mathcal{B}(X)$ such that

(b) [5p] Show that $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{B}(X)$.

Let $A, B \in \mathcal{B}(X)$. Then

$$\langle (AB)^* x, y\rangle = \langle x, ABy\rangle = \langle A^*x, By\rangle = \langle B^*A^*x, y\rangle$$

for all $x, y \in X$. Therefore $(AB)^* = B^*A^*$.

(c) [5p] Now let $u, v \in X$. Define an operator $E : X \to X$ by

$$Ef = \langle u, f\rangle\, v.$$

Calculate E^*.

Since

$$\langle E^*g, f\rangle = \langle g, Ef\rangle = \langle g, \langle u, f\rangle\, v\rangle = \langle u, f\rangle\, \langle g, v\rangle = \left\langle \overline{\langle g, v\rangle} u, f\right\rangle = \langle \langle v, g\rangle\, u, f\rangle$$

for all $f, g \in X$, we can see that E^* is given by

$$E^*f = \langle v, f\rangle\, u.$$

(d) [10p] Show that

$$\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^\perp$$

for all $A \in \mathcal{B}(X)$.

Let $A \in \mathcal{B}(X)$ and $f \in X$. Since

$$\begin{aligned} f \in \operatorname{Ker}(A^*) &\iff A^*f = 0 \\ &\iff \langle A^*f, g\rangle = 0 \quad \forall g \in X \\ &\iff \langle f, Ag\rangle = 0 \quad \forall g \in X \\ &\iff \langle f, h\rangle = 0 \quad \forall h \in \operatorname{Ran}(A) \\ &\iff f \in \operatorname{Ran}(A)^\perp, \end{aligned}$$

it follows that

$$\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^\perp.$$