OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

14.11.2012 **MAT 461 – Fonksiyonel Analiz I – Ara Sınav** N. Course

ADI SOYADI
ÖĞRENCİ NO Çözümlar
İMZA

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have 60 minutes to answer 2 questions from a choice of 3. If you choose to answer more than 2 questions, then only your best 2 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You should write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam 60 dakikadır. Sınavda 3 soru sorulmuştur. Bu sorulardan 2 tanesini seçerek cevaplayınız. 2'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 2 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	TOTAL

ÖĞRENCİ NO.

Question 1 (Banach spaces). Let X be a vector space.

(a) [5p] Give the definition of an *inner product* on X.

An inner product on X is a function $\langle \cdot , \cdot \rangle : X \times X \xrightarrow{1} \mathbb{C}$ which satisfies

1 (i) $\langle \alpha f + \beta g, h \rangle = \bar{\alpha} \langle f, h \rangle + \bar{\beta} \langle g, h \rangle$

1 (ii) $\langle f, \alpha g + \beta h \rangle = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$

$\Big\} \forall f, g, h \in X, \forall \alpha, \beta \in \mathbb{C}$.

1 (iii) $\langle f, f \rangle > 0 \quad \forall f \neq 0$

1 (iv) $\langle f, g \rangle = \overline{\langle g, f \rangle}$.

(b) [4p] Give the definition of a *Banach space*.

A complete (2), normed (2) vector space is called a Banach space.

Let $I = [0, 2] \subseteq \mathbb{R}$,

$$C(I) = \{ f : I \to \mathbb{C} : f \text{ is continuous} \}$$

and

$$\langle f, g \rangle_{L^2} = \int_0^2 \overline{f(x)} g(x) \, dx.$$

(c) [10p] Show that $\langle \cdot , \cdot \rangle_{L^2}$ is a inner product on $C(I)$.

Note first that

$$\langle \alpha f + \beta g, h \rangle_{L^2} = \int_0^2 \overline{\alpha f(x) + \beta g(x)} \, h(x) \, dx$$
$$= \bar{\alpha} \int_0^2 \overline{f(x)} \, h(x) \, dx + \bar{\beta} \int_0^2 \overline{g(x)} \, h(x) \, dx = \bar{\alpha} \langle f, h \rangle_{L^2} + \bar{\beta} \langle g, h \rangle.$$

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Moreover

$$\langle f, g \rangle_{L^2} = \int_0^2 \overline{f(x)} \, g(x) \, dx = \overline{\int_0^2 f(x) \, \overline{g(x)} \, dx} = \overline{\langle g, f \rangle_{L^2}}.$$ 2

Property (ii) follows automatically (1) from (i) & (iv). Finally, since f is continuous, if $f(x_0) \neq 0$, then $\exists$ an interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ where $f(x) \neq 0$. So

$$\langle f, f \rangle_{L^2} = \int_0^2 |f(x)|^2 \, dx > 0.$$

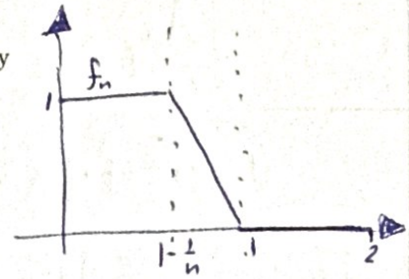
4

Therefore $\langle \cdot , \cdot \rangle_{L^2}$ is an inner product.

Define

$$\|f\|_{L^2} := \sqrt{\langle f, f\rangle_{L^2}} = \left(\int_0^2 |f(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

This is a norm by question 2. Define a sequence of functions $(f_n)_{n=1}^{\infty}$ by



$$f_n(x) = \begin{cases} 1 & 0 \le x \le 1 - \frac{1}{n}, \\ n(1-x) & 1 - \frac{1}{n} < x < 1, \\ 0 & 1 \le x \le 2. \end{cases}$$

It is easy to see that $f_n \in C(I)$ for all $n \in \mathbb{N}$.

(d) [12p] Show that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(C(I), \|\cdot\|_{L^2})$.

Let $\varepsilon > 0$. Choose $N > \varepsilon^2$ 4. Then

$$n > m > N \Rightarrow \|f_n - f_m\|_{L^2}^2 = \int_0^2 |f_n(x) - f_m(x)|^2 dx$$

$$= \int_{1-\frac{1}{m}}^{1} |f_n(x) - f_m(x)|^2 dx \quad 4$$

$$\le \int_{1-\frac{1}{m}}^{1} 1^2 \, dx = \frac{1}{m} < \frac{1}{N} < \varepsilon^2. \quad 4$$

$\Rightarrow \|f_n - f_m\|_{L^2} < \varepsilon$. Therefore f_n is a Cauchy sequence.

(e) [17p] Show that $(f_n)_{n=1}^{\infty}$ does not have a limit in $C(I)$.

Define $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 \le x \le 2. \end{cases}$ 4 Then

$$\|f_n - f\|_{L^2}^2 = \int_0^2 |f_n(x) - f(x)|^2 dx = \int_{1-\frac{1}{n}}^{1} |f_n(x) - f(x)|^2 dx \quad 4$$

$$\le \int_{1-\frac{1}{n}}^{1} 1^2 \, dx = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Therefore $f_n \xrightarrow{4} f$, but $\underbrace{f \notin C(I) \text{ because } f \text{ is not continuous}}_{5}$.
Therefore (f_n) does not have a limit in $C(I)$.

(f) [2p] Is $(C(I), \|\cdot\|_{L^2})$ a Banach space?

☐ Yes, ☑ No.

ÖĞRENCİ NO.

Question 2 (The Cauchy-Schwarz Inequality). Let X be a vector space.

(a) [~~10p~~ 5] Give the definition of a *norm* on X.

A norm on X is a function $\|\cdot\| : X \longrightarrow [0,\infty)$ ① (or $\mathbb{R}$).
such that

① (i) $\|f\| > 0 \quad \forall f \neq 0$

① (ii) $\|\alpha f\| = |\alpha| \cdot \|f\| \quad \forall f \in X, \forall \alpha \in \mathbb{C}$.

① (iii) $\|f+g\| \leq \|f\| + \|g\| \quad \forall f,g \in X$.

(+1 pt for any reasonable attempt).

Now let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Define a function $\|\cdot\| : X \to \mathbb{R}$ by

$$\|f\| = \sqrt{\langle f, f\rangle}$$

for all $f \in X$.

(b) [5p] Show that $\|f\| > 0$ for all $f \neq 0$.

By definition of an inner product, $f \neq 0 \Rightarrow \langle f,f\rangle > 0$.

So $\|f\| = \sqrt{\langle f,f\rangle} > 0$.

(c) [5p] Show that $\|\alpha f\| = |\alpha|\,\|f\|$ for all $f \in X$ and $\alpha \in \mathbb{C}$.

$\|\alpha f\|^2 = \langle \alpha f, \alpha f\rangle = \alpha\langle \alpha f, f\rangle = \alpha\bar{\alpha}\langle f,f\rangle = |\alpha|^2\langle f,f\rangle = |\alpha|^2\|f\|^2$

So $\|\alpha f\| = |\alpha| \cdot \|f\| \quad \forall \alpha \in \mathbb{C}$ and $\forall f \in X$.

Let $f, g \in X$ and define $\hat{g} := \frac{1}{\|g\|} g$.

(d) [2p] Calculate $\|\hat{g}\|$.

$\|\hat{g}\| = \left\|\frac{1}{\|g\|} g\right\| = \frac{1}{\|g\|} \cdot \|g\| = 1$.

Define $f_\parallel = \langle \hat{g}, f\rangle\,\hat{g}$ and $f_\perp = f - f_\parallel$.

(e) [~~4p~~ 6] Show that $\|f\|^2 = |\langle \hat{g}, f\rangle|^2 + \|f_\perp\|^2$.

Since $f = f_\parallel + f_\perp$ and since $f_\parallel$ and $f_\perp$ are orthogonal, it follows by Pythagoras that

$\|f\|^2 = \|f_\parallel + f_\perp\|^2 = \|f_\parallel\|^2 + \|f_\perp\|^2 = \|\langle \hat{g}, f\rangle \hat{g}\|^2 + \|f_\perp\|^2$ 5

$= |\langle \hat{g}, f\rangle|^2 \|\hat{g}\|^2 + \|f_\perp\|^2 = |\langle \hat{g}, f\rangle|^2 + \|f_\perp\|^2$.

(f) [6p] Use (e) to show that $|\langle f, \hat{g}\rangle| \le \|f\|$.

Since $\|f_\perp\| \geq 0$, it follows that

$$\|f\|^2 = |\langle \hat{g}, f\rangle|^2 + \|f_\perp\|^2 \geq |\langle \hat{g}, f\rangle|^2 = |\langle f, \hat{g}\rangle|^2.$$

Therefore

$$|\langle f, \hat{g}\rangle| \le \|f\|.$$

(g) [6p] Use (f) to show that $|\langle f, g\rangle| \le \|f\|\,\|g\|$. [This is the Cauchy-Schwarz Inequality.]

$$|\langle f, g\rangle| = |\langle f, \|g\|\hat{g}\rangle| = |\,\|g\|\langle f, \hat{g}\rangle|$$

$$= \|g\| \cdot |\langle f, \hat{g}\rangle| \le \|g\| \cdot \|f\|$$

$$= \|f\| \cdot \|g\| \qquad \text{by (f)}.$$

(h) [15p] Show that $\|\cdot\|$ is a norm on X.

By (b) and (c), all that remains is to prove the triangle inequality. 2

Since $\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$ 4

$$= \|f\|^2 + \langle f,g\rangle + \overline{\langle f,g\rangle} + \|g\|^2$$

$$\le \|f\|^2 + 2|\langle f,g\rangle| + \|g\|^2 \quad 4$$

$$\le \|f\|^2 + \underset{3}{2\|f\|\cdot\|g\|} + \|g\|^2 = (\|f\| + \|g\|)^2 \quad 1$$

it follows that $\|f+g\| \le \|f\| + \|g\|$. 1 Therefore $\|\cdot\|$ is a norm on X.

ÖĞRENCİ NO.

Question 3 (Equivalence of Norms). Let X be a vector space.

(a) [5p] Complete the following definition:

Two norms on X, $\|\cdot\|_1$ and $\|\cdot\|_2$, are called *equivalent* if and only if ...

$\exists\, m_1, m_2 > 0$ such that

$$\frac{1}{m_2}\|f\|_1 \leq \|f\|_2 \leq m_1 \|f\|_1 \quad \forall f \in X.$$

(b) [10p] Show that equivalence of norms is an equivalence relation.

Let $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$ be norms on X.

(i) Clearly $\|\cdot\|_1 \sim \|\cdot\|_1$ (choose $m_1 = m_2 = 1$).

(ii) Suppose $\|\cdot\|_1 \sim \|\cdot\|_2$. Then $\exists\, m_1, m_2 > 0$ s.t.

$$\frac{1}{m_2}\|f\|_1 \leq \|f\|_2 \leq m_1\|f\|_1 \quad \forall f \in X$$

$$\Rightarrow \quad \frac{1}{m_1}\|f\|_2 \leq \|f\|_1 \leq m_2 \|f\|_2$$

$\Rightarrow \|\cdot\|_2 \sim \|\cdot\|_1$.

(iii) Finally suppose $\|\cdot\|_1 \sim \|\cdot\|_2$ and $\|\cdot\|_2 \sim \|\cdot\|_3$. Then $\exists\, m_1, m_2, n_2, n_3 > 0$ s.t. $\frac{1}{m_2}\|f\|_1 \leq \|f\|_2 \leq m_1\|f\|_1$ and $\frac{1}{n_3}\|f\|_2 \leq \|f\|_3 \leq n_2\|f\|_2$ $\forall f \in X$. But then

$$\frac{1}{m_2 n_3}\|f\|_1 \leq \|f\|_3 \leq m_1 n_2 \|f\|_1 \quad \forall f \in X \Rightarrow \|\cdot\|_1 \sim \|\cdot\|_3.$$

So $\sim$ is an equivalence relation.

(c) [5p] Give the definition of a *separable* normed vector space.

$(X, \|\cdot\|)$ is called separable if there exists 1 a countable 2 dense 2 subset.

For the rest of this question: Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

(d) [20p] Show that

$$A \text{ is a dense subset of } (X, \|\cdot\|_1) \iff A \text{ is a dense subset of } (X, \|\cdot\|_2)$$

A is a dense subset of $(X, \|\cdot\|_1) \overset{5}{\Rightarrow} \forall \varepsilon > 0 \ \forall x \in X \ \exists a \in A$ s.t. $\|x-a\|_1 < \frac{\varepsilon}{m_1}$

5 $\Rightarrow \forall \varepsilon > 0 \ \forall x \in X \ \exists a \in A$ s.t. $\|x-a\|_2 \leq m_1 \|x-a\|_1 < \varepsilon$

5 $\Rightarrow$ A is a dense subset of $(X, \|\cdot\|_2)$.

"$\Leftarrow$" follows immediately by (b). 5

(e) [10p] Show that

$$(X, \|\cdot\|_1) \text{ is separable} \iff (X, \|\cdot\|_2) \text{ is separable}$$

$(X, \|\cdot\|_1)$ is separable $\overset{\text{def}^n}{\iff}$ 2 $\exists$ a countable subset A which is dense in $(X, \|\cdot\|_1)$

6 $\overset{(d)}{\iff} \exists$ a countable subset A which is dense in $(X, \|\cdot\|_2)$

2 $\overset{\text{def}^n}{\iff} (X, \|\cdot\|_2)$ is separable.