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MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2014.01.08 **MAT 461 – Fonksiyonel Analiz I – Final Sınavın Çözümleri** N. Course

Soru 1 (Compact Operators)**.** Let X, Y and Z be normed spaces.

(a) [4p] Give the definition of a *compact* operator $K : X \to Y$.

An operator K is called a *compact* operator iff, for all bounded sequences $(x_n) \subseteq X$ there exists a subsequence (x_{n_j}) such that $(Kx_{n_j}) \subseteq Y$ is convergent.

(b) [3p] Give the definition of a *bounded* operator $B : X \to Y$.

An operator B is called *bounded* iff $\|B\| < \infty$.

Let $\mathcal{B}(X,Y) = \{A : X \to Y : A \text{ is linear and bounded}\}$ and $\mathcal{K}(X,Y) = \{A : X \to Y : A \text{ is linear and compact}\}$.

(c) [9p] Show that $\mathcal{K}(X,Y) \subseteq \mathcal{B}(X,Y)$.
[HINT: In other words: Show that every compact operator is bounded.]

Suppose that $A \in \mathcal{K}(X,Y)$ is not bounded. Then $\forall\ n \in \mathbb{N}$, $\exists$ a unit vector u_n such that $\|Au_n\| \geq n$. Since the sequence $\{u_n\}$ is bounded and since A is compact, $\exists$ a convergent subsequence $\{Au_{n_j}\}$. But this contradicts $\|Au_n\| \geq n$. So A must be bounded.

(d) [9p] Let $K \in \mathcal{K}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$. Show that $(BK) \in \mathcal{K}(X,Z)$.

Let f_n be any bounded sequence in X. Since K is compact, $\exists$ a subsequence f_{n_j} such that Kf_{n_j} is convergent in Y. Since B is bounded (and hence continuous), it follows that BKf_{n_j} is convergent in Z. Therefore BK is compact.

Soru 2 (Orthonormal Bases)**.** Let X be a Hilbert space.

(a) [4p] Give the definition of an *orthonormal set*.

A set $\{u_j\}_{j\in J} \subset X$ is called an *orthonormal set* if and only if

$$\langle u_j, u_k \rangle = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k. \end{cases}$$

(b) [3p] Give the definition of an *orthonormal basis*.

An orthonormal set is called an *orthonormal basis* for X, iff every $f \in X$ can be written as

$$f = \sum_{j\in J} \langle u_j, f \rangle u_j.$$

For the rest of this question, let $\{u_j\}_{j\in J}$ be an orthonormal **set** in X.

(c) [8p] Suppose that

$$\|f\|^2 = \sum_{j\in J} |\langle u_j, f\rangle|^2$$

for all $f \in X$. Show that

$$\langle u_j, f\rangle = 0 \quad \forall j \in J \qquad \Longrightarrow \qquad f = 0.$$

If $\langle u_j, f\rangle = 0$ for all $j \in J$, then we have that $\|f\|^2 = \sum_{j\in J} |\langle u_j, f\rangle|^2 = 0$ which implies that $f = 0$.

(d) [10p] Now suppose that

$$\langle u_j, f\rangle = 0 \quad \forall j \in J \qquad \Longrightarrow \qquad f = 0$$

is true. Suppose that $Q \supseteq \{u_j\}_{j\in J}$ is an orthonormal set. Show that $Q = \{u_j\}_{j\in J}$.

We start by supposing that $\exists\ g \in Q$ such that $g \notin \{u_j\}_{j\in J}$. Since Q is orthonormal, g is a unit vector. But then $\langle u_j, g\rangle = 0$ for all j because Q is orthonormal. It follows by the assumption in the question that we must have $g = 0$. Contradiction.

Soru 3 (Unitary Operators)**.** Let $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ be Hilbert spaces. Define $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle_X}$ and $\|\cdot\|_Y = \sqrt{\langle\cdot,\cdot\rangle_Y}$ as usual.

(a) [5p] Give the definition of a *unitary operator*, $U : X \to Y$.

U is called *unitary* iff

(a) U is a bijection; and

(b) $\langle Uf, Ug\rangle_Y = \langle f, g\rangle_X$ for all $f, g \in X$.

(b) [5p] Suppose that the (linear) operator $A : X \to Y$ satisfies $\|Af\|_Y = \|f\|_X$ for all $f \in X$. Show that A is an injection.

Let $f, g \in X$. Suppose $Af = Ag$. Then

$$0 = \|Af - Ag\| = \|A(f - g)\| = \|f - g\|.$$

Therefore $f = g$. This proves that A is an injection.

(c) [5p] Suppose that the (linear) operator $A : X \to Y$ is a surjection and satisfies $\|Af\|_Y = \|f\|_X$ for all $f \in X$. Show that A is unitary.

Follows by part (b) and the Parallelogram Law.

Now suppose that $X = Y$.

(d) [10p] Suppose $U : X \to X$ is unitary and $M \subseteq X$. Show that

$$U(M^\perp) = (UM)^\perp.$$

First, note that since U is a bijection, $\forall k \in X$, $\exists f \in X$ such that $k = Uf$. Now

$$\begin{aligned} Uf \in U(M^\perp) &\iff f \in M^\perp \\ &\iff \langle g, f \rangle = 0 \ \forall g \in M \\ &\iff \langle Ug, Uf \rangle = 0 \ \forall g \in M \quad \text{(since } U \text{ is unitary)} \\ &\iff \langle h, Uf \rangle = 0 \ \forall h \in UM \\ &\iff Uf \in (UM)^\perp. \end{aligned}$$

Therefore $U(M^\perp) = (UM)^\perp$.

Soru 4 (Eigenvalues and Eigenvectors)**.** Let X be a Hilbert space.

(a) [4p] Give the definition of a *symmetrical* operator.

An operator $A : \mathfrak{D}(A) \subseteq X \to X$ is called *symmetrical* iff its domain is dense and

$$\langle g, Af \rangle = \langle Ag, f \rangle$$

for all $f, g \in \mathfrak{D}(A)$.

(b) [4p] Give the definitions of an *eigenvalue* and an *eigenvector*, of an operator $A : \mathfrak{D}(A) \subseteq X \to X$.

$\lambda \in \mathbb{C}$ is called an *eigenvalue* of A iff $\exists\ u \in \mathfrak{D}(A)$ such that

$$Au = \lambda u.$$

The vector u is called an *eigenvector* corresponding to λ.

For the rest of this question, suppose that:

- A is a symmetrical operator;
- $\lambda \in \mathbb{C}$ is an eigenvalue of A;
- u is an eigenvector of A corresponding to λ;
- $\|u\| = 1$;
- $\mu \in \mathbb{C}$ is an eigenvalue of A;
- v is an eigenvector of A corresponding to μ;
- $\|v\| = 1$;
- $\lambda \neq \mu$.

(c) [9p] Show that $\lambda \in \mathbb{R}$.
[HINT: Remember that $\langle u, u \rangle = 1$. Use this to show that $\lambda = \overline{\lambda}$.]

Since
$$\lambda = \lambda \langle u, u \rangle = \langle u, \lambda u \rangle = \langle u, Au \rangle = \langle Au, u \rangle = \langle \lambda u, u \rangle = \overline{\lambda} \langle u, u \rangle = \overline{\lambda},$$
it follows that $\lambda \in \mathbb{R}$.

(d) [6p] Show that

$$(\lambda - \mu)\langle u, v\rangle = 0.$$

$$\begin{aligned}(\lambda - \mu)\langle u, v\rangle &= \lambda\langle u, v\rangle - \mu\langle u, v\rangle \\ &= \langle \lambda u, v\rangle - \langle u, \mu v\rangle \quad (\text{since } \lambda \in \mathbb{R}) \\ &= \langle Au, v\rangle - \langle u, Av\rangle \\ &= 0\end{aligned}$$

because A is symmetrical.

(e) [2p] Show that u is orthogonal to v.

Since $\lambda \neq \mu$, it follows from (d) that $\langle u, v\rangle = 0$. Therefore u is orthogonal to v.

Soru 5 (Adjoints of Operators)**.** Let X be a Hilbert space and let $\mathcal{B}(X) = \{A : X \to X : A$ is linear and bounded$\}$.

(a) [5p] Let $A \in \mathcal{B}(X)$ be an operator. Give the definition of the *adjoint*, A^*, of A.

The *adjoint* of A is the unique operator $A^* \in \mathcal{B}(X)$ such that

$$\langle A^* f, g\rangle = \langle f, Ag\rangle$$

for all $f, g \in X$.

(b) [5p] Let $A \in \mathcal{B}(X)$. Show that $A^{**} = A$.

Since

$$\langle f, Ag\rangle = \langle A^* f, g\rangle = \overline{\langle g, A^* f\rangle} = \overline{\langle A^{**} g, f\rangle} = \langle f, A^{**} g\rangle$$

for all $f, g \in X$, it follows that $A = A^{**}$.

Let $u, v \in X$. Let $A : X \to X$ be an operator defined by

$$Af = \langle u, f\rangle v.$$

(c) [5p] Show that A is bounded.

$\|Af\| = \|\langle u, f\rangle v\| = |\langle u, f\rangle| \, \|v\| \leq \|u\| \, \|f\| \, \|v\|$ by Cauchy-Schwarz. Therefore $\|A\| \leq \|u\| \, \|v\|$.

(d) [5p] Calculate $\|A\|$.

By Cauchy-Schwarz, $\|Af\| = \|u\| \, \|f\| \, \|v\|$ iff f is parallel to u. Therefore $\|A\| = \|u\| \, \|v\|$.

(e) [5p] Calculate the adjoint of A.

Since $\langle A^* g, f\rangle = \langle g, Af\rangle = \langle g, \langle u, f\rangle v\rangle = \langle u, f\rangle \langle g, v\rangle = \overline{\langle v, g\rangle} \langle u, f\rangle = \langle \langle v, g\rangle u, f\rangle$, it follows that $A^* g = \langle v, g\rangle u$.