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MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2013.11.12 **MAT 461 – Fonksiyonel Analiz I – Ara Sınavın Çözümleri** N. Course

Soru 1 (Banach Spaces)**.** Let X be a vector space

(a) [10p] Give the definition of a *norm* on X.

A *norm* is a function $\|\cdot\| : X \to \mathbb{R}$ $\boxed{2}$ which satisfies

(i) $\|f\| > 0$ for all $f \in X$, $f \neq 0$; $\boxed{3}$

(ii) $\|\alpha f\| = |\alpha| \, \|f\|$ for all $f \in X$, $\alpha \in \mathbb{C}$ (or $\alpha \in \mathbb{R}$ if X is a real vector space); $\boxed{3}$

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$. $\boxed{2}$

Consider the set

$$\ell^\infty(\mathbb{N}) := \{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_\infty < \infty\}$$

where

$$\|a\|_\infty := \sup_{j \in \mathbb{N}} |a_j| \, .$$

(b) [10p] Show that $\ell^\infty(\mathbb{N})$ is a vector space.

If $a = (a_j)_{j=1}^\infty$ and $b = (b_j)_{j=1}^\infty$ are sequences in $\ell^\infty(\mathbb{N})$ and if $\lambda \in \mathbb{C}$, then $\|a + \lambda b\|_\infty = \sup_j |a_j + \lambda b_j| \leq \sup_j |a_j| + |\lambda| \sup_j |b_j| = \|a\|_\infty + |\lambda| \, \|b\|_\infty < \infty$, so $a + \lambda b \in \ell^\infty(\mathbb{N})$.

(c) [10p] Show that $\|\cdot\|_\infty$ is a norm on $\ell^\infty(\mathbb{N})$.

Clearly $\|a\|_\infty > 0$ for all $a \in \ell^\infty(\mathbb{N})$, $a \neq 0$ (i.e. not all $a_n = 0$). $\boxed{5}$ Positive homogeneity and the triangle inequality were shown in part (a).$\boxed{5}$

(d) [5p] Give the definition of a *Banach space*.

A *Banach space* is a complete $\boxed{2}$ normed $\boxed{2}$ vector space. $\boxed{1}$

(e) [15p] Show that $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is a Banach space.

We must show that $\ell^\infty(\mathbb{N})$ is complete: Let $a^n = (a_j^n)_{j=1}^\infty$ be a Cauchy sequence in $\ell^\infty(\mathbb{N})$ (i.e. a sequence of sequences). Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $m, n > N \implies \|a^m - a^n\|_\infty < \varepsilon \implies \left|a_j^m - a_j^n\right| < \varepsilon$ for all j. For each j, $(a_j^n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, there exists a limit $a_j = \lim_{n\to\infty} a_j^n$. We must prove that $a = (a_j)$ is in $\ell^\infty(\mathbb{N})$.

Since $\left|a_j^m - a_j^n\right| < \varepsilon$ for all $m, n > N$ and all j, we must also have that $\left|a - a_j^n\right| \leq \varepsilon$ for all $n > N$ and all j. It follows that $\|a - a^n\|_\infty = \sup_j \left|a_j - a_j^n\right| \leq \varepsilon$ for all $n > N$ and so $\|a\|_\infty \leq \|a - a^n\|_\infty + \|a^n\|_\infty < \infty$. Therefore $a \in \ell^\infty(\mathbb{N})$ and so $\ell^\infty(\mathbb{N})$ is complete.

Soru 2 (Separable Spaces)**.**

(a) [10p] Give the definition of *separable*.

A normed space X is called *separable* iff it contains a countable dense subset.

Consider the Banach space

$$\ell^\infty(\mathbb{N}) := \{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_\infty < \infty\}, \qquad \|a\|_\infty := \sup_{j\in\mathbb{N}} |a_j|.$$

Define

$$S := \{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : a_j \in \{0,1\}\ \forall j\} \subseteq \ell^\infty(\mathbb{N}).$$

For example, the sequence $(1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots)$ is in S.

(b) [10p] Let $a, b \in S$, $a \neq b$. Calculate $\|a - b\|_\infty$.

Since $a, b \in S$ we know that $|a_j - b_j| = 0$ or 1 for all j. If $a \neq b$, then $\exists\, k$ such that $a_k \neq b_k$. Therefore $|a_k - b_k| = 1$ and so

$$\|a - b\|_\infty = 1.$$

(c) [10p] Show that S is not countable.

Suppose that S is countable. Then we can label every element of S as $a^1, a^2, a^3, a^4, \ldots$.

Now define a new sequence b as follows: if $a_j^j = 1$ then define $b_j = 0$, otherwise define $b_j = 1$. Then for every n, $b \neq a^n$ (because $b_n \neq a_n^n$) so b is not in our list $a^1, a^2.a^3, a^4, \ldots$. However, every term in the sequence of b is either 1 or 0, so b must be in S. Contradiction.

Therefore S is uncountable.

(d) [20p] Show that $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is <u>not</u> separable.

Finally suppose that $P = \{p^1, p^2, p^3, \ldots\}$ is a countable, dense subset of $\ell^\infty(\mathbb{N})$ and let $0 < \varepsilon < \frac{1}{10}$. Then for every $a \in S$ there must exists some $p \in P$ such that $\|a - p\|_\infty < \varepsilon$. But since $a, b \in S$, $a \neq b \implies \|a - b\|_\infty = 1$, each $p \in P$ can only be close to at most one element in S. But S is uncountable so P must be uncountable as well. Contradiction.

Therefore $\ell^\infty(\mathbb{N})$ is not separable.

Soru 3 (Inner Products)**.** Let X be a vector space.

(a) [10p] Give the definition of an *inner product*.

An *inner product* is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ $\boxed{2}$ such that

(i) $\langle \alpha f + \beta g, h \rangle = \overline{\alpha} \langle f, h \rangle + \overline{\beta} \langle g, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$; $\boxed{2}$

(ii) $\langle f, \alpha g + \beta h \rangle = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$; $\boxed{2}$

(iii) $\langle f, f \rangle > 0$ for all $f \in X$, $f \neq 0$; $\boxed{2}$ and

(iv) $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in X$. $\boxed{2}$

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\|f\| := \sqrt{\langle f, f \rangle}$ for all $f \in X$.

(b) [10p] Suppose that $u, v \in X$ are orthogonal ($u \perp v$). Show that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 .$$

Since u and v are orthogonal, we know that $\langle u, v \rangle = 0$. Thus we have that $\langle v, u \rangle = \overline{\langle u, v \rangle} = \overline{0} = 0$ also $\boxed{5}$. Therefore

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + 0 + 0 + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 . \boxed{5} \end{aligned}$$

(c) [10p] Show that

$$\|f + g\|^2 - \big(\|f\| + \|g\| \big)^2 = 2 \operatorname{Re} \langle f, g \rangle - 2 \|f\| \|g\|$$

for all $f, g \in X$.

$$\begin{aligned} \|f + g\|^2 - \big(\|f\| + \|g\| \big)^2 &= \Big(\|f\|^2 + \langle f, g \rangle + \langle g, f \rangle + \|g\|^2 \Big) - \Big(\|f\|^2 + 2 \|f\| \|g\| + \|g\|^2 \Big) \\ &= \langle f, g \rangle + \langle g, f \rangle - 2 \|f\| \|g\| \\ &= \langle f, g \rangle + \overline{\langle f, g \rangle} - 2 \|f\| \|g\| \\ &= 2 \operatorname{Re} \langle f, g \rangle - 2 \|f\| \|g\| \end{aligned}$$

(d) [20p] Let $f, g \in X$ and $g \neq 0$. Show that

$$\|f + g\| = \|f\| + \|g\| \qquad \Longleftrightarrow \qquad f = \alpha g \quad \text{for some } \alpha \in \mathbb{R},\ \alpha \geq 0.$$

[HINT: Use the Cauchy-Schwarz Inequality.]

By part (b) we can see that $\|f + g\| = \|f\| + \|g\|$ if and only if

$$\operatorname{Re} \langle f, g \rangle = \|f\| \|g\| .$$

But

$$\operatorname{Re} \langle f, g \rangle \leq |\langle f, g \rangle| \leq \|f\| \|g\|$$

by the Cauchy-Schwarz Inequality. The second "$\leq$" is an "$=$" if and only if $f = \alpha g$ for some $\alpha \in \mathbb{C}$ (by the Cauchy-Schwarz Inequality). Since $\langle \alpha g, g \rangle = \bar{\alpha} \|g\|^2$, the first "$\leq$" is an "$=$" if and only if $\alpha \in \mathbb{R}$ and $\alpha \geq 0$.