OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2015.01.08 **MAT461 Fonksiyonel Analiz I – Final Sınavı** N. Course

ADI:	Ö R N E K T İ R
SOYADI:	S A M P L E
ÖĞRENCİ NO:	
İMZA:	

Süre: **120** dk.

Sınav sorularından **4** tanesini seçerek cevaplayınız.

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have **120** minutes to answer **4** questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You must show your working for all questions.
5. Write your student number on every page.
6. This exam contains 12 pages. Check to see if any pages are missing.
7. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the first 20 minutes, or in the final 10 minutes, of the exam.
8. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
9. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
10. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam **120** dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan **4** tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Sonuca ulaşmak için yaptığınız işlemleri ayrıntılarıyla gösteriniz.
5. Öğrenci numaranızı her sayfaya yazınız.
6. Sınav 12 sayfadan oluşmaktadır. Lütfen eksik sayfa olup olmadığını kontrol edin.
7. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın ilk 20 dakikası ve son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
8. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
9. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
10. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	4	5	TOPLAM
25	25	25	25	25	100

Notation:

$$C([a,b]) = \{f : [a,b] \to \mathbb{C} : f \text{ is continuous }\}$$
$$C^1([a,b]) = \{f : [a,b] \to \mathbb{C} : f \text{ and } f' \text{ are continuous }\}$$
$$C^\infty([a,b]) = \{f : [a,b] \to \mathbb{C} : \tfrac{d^n f}{dx^n} \text{ exists and is continuous } \forall n\}$$
$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$
$$\|f\|_{\infty,1} = \|f\|_\infty + \|f'\|_\infty$$

$$\ell^p(\mathbb{N}) = \Big\{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \sum_{j=1}^\infty |a_j|^p < \infty\Big\}$$
$$\|a\|_p = \Big(\sum_{j=1}^\infty |a_j|^p\Big)^{\frac{1}{p}}$$
$$\ell^\infty(\mathbb{N}) = \Big\{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \sup_j |a_j| < \infty\Big\}$$
$$\|a\|_\infty = \sup_j |a_j|$$

$$\mathcal{L}^2_{cont}([a,b]) = \big(C([a,b]), \langle \cdot, \cdot \rangle_{L^2}\big)$$
$$\langle f, g \rangle_{L^2} = \int_a^b \overline{f(x)} g(x)\, dx$$

$$\mathcal{B}(X,Y) = \{A : X \to Y : A \text{ is linear and bounded}\}$$
$$\mathcal{B}(X) = \mathcal{B}(X,X)$$
$$\mathcal{K}(X,Y) = \{A : X \to Y : A \text{ is linear and compact}\}$$

$$\overline{x+iy} = x - iy$$
$$A^* = \text{adjoint of } A$$
$$\text{Ker}(A) = \text{kernal of } A = \{f \in X : Af = 0\}$$
$$\text{Ran}(A) = \text{range of } A = \{Af : f \in X\}$$
$$M^\perp = \text{orthogonal complement of } M$$

$$\wedge = \text{“and”}$$
$$\vee = \text{“or”}$$

Soru 1 (Operators defined via forms) Let X be a Hilbert space.

(a) [4p] Give the definition of a *sesquilinear form* on X.

(b) [1p] Which important theorem/lemma from this course says the following: *"Let X be a Hilbert space and let $l \in X^*$. Then $\exists$ a unique vector $h \in X$ such that $l(f) = \langle h, f \rangle$ for all $f \in X$."*?

- ☐ The Cauchy-Schwarz Inequality;
- ☐ The Lax-Milgram Theorem;
- ☐ The Riesz Representation Theorem;
- ☐ The Parallelogram Law;
- ☐ The Heine-Borel Theorem;
- ☐ Zorn's Lemma;
- ☐ The BLT Theorem;
- ☐ The Arzelà-Ascoli Theorem.

(c) [4p] Let $f \in X$. Calculate

$$\sup_{\substack{\|g\|=1 \\ g \in X}} |\langle g, f \rangle| .$$

Now let $s : X \times X \to \mathbb{C}$ be a bounded sesquilinear form. For each $g \in X$, we can define a map $l_g : X \to \mathbb{C}$ by

$$l_g(f) := \overline{s(f, g)}.$$

Since s is sesquilinear, it is easy to see that l_g is linear.

(d) [5p] Show that

$$l_{g+\lambda v}(f) = l_g(f) + \overline{\lambda} l_v(f)$$

for all $f, g, v \in X$ and all $\lambda \in \mathbb{C}$.

By the result quoted in part (b); we know that, for each $g \in X$, there exists a unique vector $h_g \in X$ such that

$$l_g(\cdot) = \langle h_g, \cdot \rangle .$$

Define an operator $A : X \to X$ by

$$Ag = h_g.$$

(e) [4p] Show that A is linear.

[HINT: Use part (d).]

Since s is bounded, we have that $\|Af\|^2 = \langle Af, Af \rangle = s(Af, f) \leq C \|Af\| \|f\|$, for some constant $C \geq 0$. So $\|Af\| \leq C \|f\|$ and hence A is bounded.

(f) [7p] Show that

$$\|A\| = \sup_{\substack{\|f\|=\|g\|=1 \\ f,g \in X}} |s(f, g)| .$$

[HINT: Use your answer to part (c).]

Soru 2 (The Proof of the BLT Theorem) Let X be a normed space. Let Y be a Banach space.

(a) [3p] Give the definition of the *Operator Norm*.

(b) [2p] Give the definition of a *bounded* operator.

Now suppose that

- $\mathfrak{D}(A) \subseteq X$ is a dense subset;
- $A : \mathfrak{D}(A) \to Y$ is a linear operator;
- A is bounded;
- $v \in X$;
- $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are Cauchy sequences in $\mathfrak{D}(A)$;
- $\lim_{n\to\infty} f_n = v$;
- $\lim_{n\to\infty} g_n = v$.

(c) [5p] Show that

$$\lim_{n\to\infty} Af_n = \lim_{n\to\infty} Ag_n.$$

[HINT: If $v \in \mathfrak{D}(A)$, then this is easy: $\lim_{n\to\infty} Af_n = Av = \lim_{n\to\infty} Ag_n$ because A is continuous. However, if $v \in X \setminus \mathfrak{D}(A)$, then Av is undefined.]

Now we can define a new map $\overline{A} : X \to Y$ as follows: For all $f \in X$, let $(f_n)_{n=1}^{\infty} \subseteq \mathfrak{D}(A)$ be a Cauchy sequence such that $f_n \to f$ as $n \to \infty$ (remember that $\mathfrak{D}(A)$ is dense in X, so we can always find such a Cauchy sequence). Then define

$$\overline{A}f := \lim_{n\to\infty} Af_n.$$

(d) [5p] Show that if $f \in \mathfrak{D}(A)$, then $\overline{A}f = Af$.

(e) [5p] Show that $\overline{A}$ is linear.

(f) [5p] Show that $\|\overline{A}\| = \|A\|$.

Soru 3 (The Spectral Theorem for Compact Symmetrical Operators) Let X and Y be normed spaces.

(a) [5p] Give the definition of a *compact* operator $K : X \to Y$.

Now suppose that

- X is a Hilbert space;
- $A : X \to X$;
- $A \in \mathcal{K}(X)$;
- A is symmetrical;
- $\|A\| \neq 0$;
- $\alpha := \|A\|$.

(b) [4p] Show that

$$\|A\|^2 = \sup_{\substack{\|f\|=1 \\ f \in X}} \left\langle f, A^2 f \right\rangle .$$

By part (b), $\exists$ a sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \left\langle u_n, A^2 u_n \right\rangle = \alpha^2.$$

(c) [3p] Show that $\exists$ a subsequence $\{u_{n_j}\}_{j=1}^{\infty} \subseteq \{u_n\}_{n=1}^{\infty}$ such that $A^2 u_{n_j}$ converges as $j \to \infty$.

(d) [3p] Show that

$$\left\|A^2 u_{n_j}\right\| \leq \alpha^2$$

for all j.

(e) [10p] Show that

$$\lim_{j\to\infty} A^2 u_{n_j} = \lim_{j\to\infty} \alpha^2 u_{n_j}.$$

[HINT: $\left\|(A^2-\alpha^2)u_{n_j}\right\|^2 = \left\langle A^2 u_{n_j} - \alpha^2 u_{n_j}, A^2 u_{n_j} - \alpha^2 u_{n_j}\right\rangle = \left\|A^2 u_{n_j}\right\|^2 - ? + ?.$]

Soru 4 (Inner Products and the Parallelogram Law)

(a) [5p] Give the definition of a *total* set.

Definition: An *inner product* is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that

(i) $\langle \alpha f + \beta g, h \rangle = \overline{\alpha} \langle f, h \rangle + \overline{\beta} \langle g, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$;

(ii) $\langle f, \alpha g + \beta h \rangle = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$;

(iii) $\langle f, f \rangle > 0$ for all $f \in X$, $f \neq 0$; and

(iv) $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in X$.

(b) [5p] Show that condition (ii) is not necessary in the definition of an inner product. Precisely, show that

$$\Big(\text{(i)} \wedge \text{(iv)} \Big) \implies \text{(ii)}.$$

Let X be an inner product space.

(c) [5p] Let $u, v \in X$. Suppose that $\langle u, x \rangle = \langle v, x \rangle$ for all $x \in X$. Show that $u = v$.

Now let $X = \mathbb{R}^n$ and define a norm $\|\cdot\|_1 : X \to \mathbb{R}$ by

$$\|x\|_1 := \sum_{j=1}^{n} |x_j|$$

for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. [You do not need to prove that $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$.]

(d) [10p] Show that $\nexists$ an inner product $\langle \cdot, \cdot \rangle_1$ such that

$$\|f\|_1 = \sqrt{\langle f, f \rangle_1}$$

for all $f \in \mathbb{R}^n$.

Soru 5 (Cauchy Sequences and Closed Subspaces) Let $(X, \|\cdot\|_X)$ be a normed space.

(a) [4p] Give the definition of a *Cauchy sequence* in X.

Consider the vector space

$$\ell^2(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \sum_{j=1}^{\infty} |a_j|^2 < \infty \right\}$$

with the inner product

$$\langle f, g \rangle_2 := \sum_{j=1}^{\infty} \overline{f_j} g_j$$

and the norm $\|f\|_2 := \sqrt{\langle f, f \rangle_2}$. Define

$$S := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \exists N \in \mathbb{N} \text{ such that } a_n = 0 \ \forall n > N \right\}.$$

(b) [7p] Show that S is a subspace of $\ell^2(\mathbb{N})$.
[HINT: The question says "sub*space*", not "sub*set*".]

Now define a sequence $\{f^n\}_{n=1}^{\infty} \subseteq S$ by

$$f_j^n := \begin{cases} 2^{\frac{1-j}{2}} & j \leq n \\ 0 & j > n. \end{cases}$$

For example,

$$f^5 = \big(1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2^{\frac{3}{2}}}, \frac{1}{4}, 0, 0, 0, 0, 0, \dots\big).$$

(c) [7p] Show that $\{f^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\ell^2(\mathbb{N})$.

(d) [7p] Show that S is not closed.