2015.01.08 **MAT461 Fonksiyonel Analiz I – Final Sınavın Çözümleri** N. Course

Soru 1 (Operators defined via forms)**.** Let X be a Hilbert space.

(a) [4p] Give the definition of a *sesquilinear form* on X.

A sesquilinear form is a function $s : X \times X \to \mathbb{C}$ such that

(a) $s(\alpha f + \beta g, h) = \bar{\alpha} s(f,h) + \bar{\beta} s(g,h)$; and

(b) $s(f, \alpha g + \beta h) = \alpha s(f,g) + \beta s(f,h)$

for all $f, g \in X$ and for all $\alpha, \beta \in \mathbb{C}$.

(b) [1p] Which important theorem/lemma from this course says the following: *"Let X be a Hilbert space and let $l \in X^*$. Then $\exists$ a unique vector $h \in X$ such that $l(f) = \langle h, f \rangle$ for all $f \in X$."*?

- ☐ The Cauchy-Schwarz Inequality;
- ☐ The Lax-Milgram Theorem;
- ✓ The Riesz Representation Theorem;
- ☐ The Parallelogram Law;
- ☐ The Heine-Borel Theorem;
- ☐ Zorn's Lemma;
- ☐ The BLT Theorem;
- ☐ The Arzelà-Ascoli Theorem.

(c) [4p] Let $f \in X$. Calculate

$$\sup_{\substack{\|g\|=1 \\ g \in X}} |\langle g, f \rangle| .$$

Cauchy-Schwarz states that $|\langle g, f \rangle| \leq \|g\| \|f\|$ with "=" iff f is parallel to g. Therefore $\sup_{\|g\|=1} |\langle g, f \rangle| = \sup_{\|g\|=1} \|g\| \|f\| = \|f\|$.

Now let $s : X \times X \to \mathbb{C}$ be a bounded sesquilinear form. For each $g \in X$, we can define a map $l_g : X \to \mathbb{C}$ by $l_g(f) := \overline{s(f,g)}$. Since s is sesquilinear, it is easy to see that l_g is linear.

(d) [5p] Show that

$$l_{g+\lambda v}(f) = l_g(f) + \overline{\lambda} l_v(f)$$

for all $f, g, v \in X$ and all $\lambda \in \mathbb{C}$.

We have that

$$l_{g+\lambda y}(f) = \overline{s(f, g+\lambda y)} = \overline{s(f,g) + \lambda s(f,y)} = \overline{s(f,g)} + \overline{\lambda s(f,y)} = l_g(f) + \overline{\lambda} l_y(f).$$

By the result quoted in part (b); we know that, for each $g \in X$, there exists a unique vector $h_g \in X$ such that $l_g(\cdot) = \langle h_g, \cdot \rangle$. Define an operator $A : X \to X$ by $Ag = h_g$.

(e) [4p] Show that A is linear. [HINT: Use part (d).]

(This question, and the the next one, are two of those things for which I said "you prove this" in class, so you have no excuse not to have thought about how to answer this question.)
Since

$$\begin{aligned}\langle A(g+\lambda y), f\rangle &= \langle h_{g+\lambda y}, f\rangle = l_{g+\lambda y}(f) = l_g(f) + \overline{\lambda} l_y(f) \\ &= \langle h_g, f\rangle + \overline{\lambda}\,\langle h_y, f\rangle = \langle h_g + \lambda h_y, f\rangle \\ &= \langle Ag + \lambda Ay, f\rangle\end{aligned}$$

for all $f \in X$, we have that $A(g+\lambda y) = Ag + \lambda Ay$.

Since s is bounded, we have that $\|Af\|^2 = \langle Af, Af\rangle = s(Af, f) \leq C\,\|Af\|\,\|f\|$, for some constant $C \geq 0$. So $\|Af\| \leq C\,\|f\|$ and hence A is bounded.

(f) [7p] Show that

$$\|A\| = \sup_{\substack{\|f\|=\|g\|=1 \\ f,g\in X}} |s(f,g)|\,.$$

[HINT: Use your answer to part (c).]

(Again, no excuses for not getting this one.)
Clearly

$$\|Af\| = \sup_{\|g\|=1} |\langle g, Af\rangle| = \sup_{\|g\|=1} |s(g,f)| = \sup_{\|g\|=1} \|f\| \left| s\left(g, \frac{f}{\|f\|}\right)\right|.$$

Therefore $\|A\| = \sup_{\|f\|=\|g\|=1} |s\,(g,f)|$.

Soru 2 (The Proof of the BLT Theorem)**.** Let X be a normed space. Let Y be a Banach space.

(a) [3p] Give the definition of the *Operator Norm*.

The operator norm of $A : \mathfrak{D}(A) \subseteq X \to Y$ is defined to be

$$\|A\| := \sup_{\substack{f\in\mathfrak{D}(A) \\ \|f\|_X=1}} \|Af\|_Y\,.$$

(b) [2p] Give the definition of a *bounded* operator.

A is called a *bounded operator* if $\|A\| < \infty$.

Now suppose that

- $\mathfrak{D}(A) \subseteq X$ is a dense subset;
- $A : \mathfrak{D}(A) \to Y$ is a linear operator;
- A is bounded;
- $v \in X$;
- $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ are Cauchy sequences in $\mathfrak{D}(A)$;
- $\lim_{n\to\infty} f_n = v$;
- $\lim_{n\to\infty} g_n = v$.

(c) [5p] Show that

$$\lim_{n\to\infty} Af_n = \lim_{n\to\infty} Ag_n.$$

[HINT: If $v \in \mathfrak{D}(A)$, then this is easy: $\lim_{n\to\infty} Af_n = Av = \lim_{n\to\infty} Ag_n$ because A is continuous. However, if $v \in X \setminus \mathfrak{D}(A)$, then Av is undefined.]

Clearly

$$\|Af_n - Ag_n\| = \|A(f_n - g_n)\| \leq \|A\|\,\|f_n - g_n\| \to 0$$

as $n \to \infty$.

Now we can define a new map $\overline{A} : X \to Y$ as follows: For all $f \in X$, let $(f_n)_{n=1}^{\infty} \subseteq \mathfrak{D}(A)$ be a Cauchy sequence such that $f_n \to f$ as $n \to \infty$ (remember that $\mathfrak{D}(A)$ is dense in X, so we can always find such a Cauchy sequence). Then define

$$\overline{A}f := \lim_{n\to\infty} Af_n.$$

(d) [5p] Show that if $f \in \mathfrak{D}(A)$, then $\overline{A}f = Af$.

Just choose the constant sequence $f_n := f$ for all n. Then clearly $\overline{A}f = \lim_{n\to\infty} Af_n = \lim_{n\to\infty} Af = Af$.

(e) [5p] Show that $\overline{A}$ is linear.

This follows from the continuity of vector addition and scalar multiplication:

If $f_n \to f$, $g_n \to g$ (f_n and g_n Cauchy sequences) and $\lambda \in \mathbb{C}$, then we know that $f_n + \lambda g_n \to f + \lambda g$. Hence

$$\overline{A}(f + \lambda g) = \lim_{n\to\infty} A(f_n + \lambda g_n) = \lim_{n\to\infty} Af_n + \lambda Ag_n = \lim_{n\to\infty} Af_n + \lambda \lim_{n\to\infty} Ag_n = \overline{A}f + \lambda\overline{A}g.$$

(f) [5p] Show that $\|\overline{A}\| = \|A\|$.

This follows from the continuity of norms: Since

$$\left\|\overline{A}f\right\| = \left\|\lim_{n\to\infty} Af_n\right\| = \lim_{n\to\infty} \|Af_n\| \leq \lim_{n\to\infty} \|A\| \, \|f_n\| = \|A\| \, \|f\| \,,$$

we have that $\|\overline{A}\| \leq \|A\|$.
The "$\geq$" follows immediately from part (d).

Soru 3 (The Spectral Theorem for Compact Symmetrical Operators)**.** Let X and Y be normed spaces.

(a) [5p] Give the definition of a *compact* operator $K : X \to Y$.

An operator K is called a *compact* operator iff, for all bounded sequences $(x_n) \subseteq X$ there exists a subsequence (x_{n_j}) such that $(Kx_{n_j}) \subseteq Y$ is convergent.

Now suppose that

- X is a Hilbert space;
- $A : X \to X$;
- $A \in \mathcal{K}(X)$;
- A is symmetrical;
- $\|A\| \neq 0$;
- $\alpha := \|A\|$.

(b) [4p] Show that

$$\|A\|^2 = \sup_{\substack{\|f\|=1 \\ f\in X}} \left\langle f, A^2 f \right\rangle.$$

Clearly

$$\|A\|^2 = \sup_{\|f\|=1, f\in X} \|Af\|^2 = \sup_{\|f\|=1, f\in X} \langle Af, Af\rangle = \sup_{\|f\|=1, f\in X} \left\langle f, A^2 f\right\rangle$$

because A is symmetrical.

By part (b), $\exists$ a sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \langle u_n, A^2 u_n \rangle = \alpha^2.$$

(c) [3p] Show that $\exists$ a subsequence $\{u_{n_j}\}_{j=1}^{\infty} \subseteq \{u_n\}_{n=1}^{\infty}$ such that $A^2 u_{n_j}$ converges as $j \to \infty$.

Since A is compact, we have that A^2 is also compact. Since u_n is a bounded sequence, this then follows straight away.

(d) [3p] Show that

$$\left\| A^2 u_{n_j} \right\| \leq \alpha^2$$

for all j.

Clearly

$$\left\| A^2 u_{n_j} \right\| \leq \|A\| \left\| A u_{n_j} \right\| \leq \|A\| \, \|A\| \left\| u_{n_j} \right\| = \alpha^2.$$

(e) [10p] Show that

$$\lim_{j\to\infty} A^2 u_{n_j} = \lim_{j\to\infty} \alpha^2 u_{n_j}.$$

[HINT: $\left\| (A^2 - \alpha^2) u_{n_j} \right\|^2 = \left\langle A^2 u_{n_j} - \alpha^2 u_{n_j}, A^2 u_{n_j} - \alpha^2 u_{n_j} \right\rangle = \left\| A^2 u_{n_j} \right\|^2 - ? + ?.$]

Since

$$\begin{aligned}
\left\| (A^2 - \alpha^2) u_{n_j} \right\|^2 &= \left\langle A^2 u_{n_j} - \alpha^2 u_{n_j}, A^2 u_{n_j} - \alpha^2 u_{n_j} \right\rangle \\
&= \left\| A^2 u_{n_j} \right\|^2 - 2\alpha^2 \left\langle u_{n_j}, A u_{n_j} \right\rangle + \alpha^4 \\
&\leq \alpha^4 - 2\alpha^2 \left\langle u_{n_j}, A^2 u_{n_j} \right\rangle + \alpha^4 \\
&= 2\alpha^2 \left(\alpha^2 - \left\langle u_{n_j}, A^2 u_{n_j} \right\rangle \right) \\
&\to 0 \qquad \boxed{6}
\end{aligned}$$

by part (b), we have that $\lim_{j\to\infty} A^2 u_{n_j} = \lim_{j\to\infty} \alpha^2 u_{n_j}$.

Soru 4 (Inner Products and the Parallelogram Law)**.**

(a) [5p] Give the definition of a *total* set.

A set whose span is dense is called *total*.

Definition: An *inner product* is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that

(i) $\langle \alpha f + \beta g, h \rangle = \overline{\alpha} \langle f, h \rangle + \overline{\beta} \langle g, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$;

(ii) $\langle f, \alpha g + \beta h \rangle = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$;

(iii) $\langle f, f \rangle > 0$ for all $f \in X$, $f \neq 0$; and

(iv) $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in X$.

(b) [5p] Show that condition (ii) is not necessary in the definition of an inner product. Precisely, show that

$$\Big(\text{(i)} \wedge \text{(iv)} \Big) \implies \text{(ii)}.$$

(An easy question – shame on you if you can't get this one.)

Let $f, g, h \in X$ and $\alpha, \beta \in \mathbb{C}$. Since

$$\langle f, \alpha g + \beta h \rangle = \overline{\langle \alpha g + \beta h, f \rangle} = \overline{\overline{\alpha} \langle g, f \rangle + \overline{\beta} \langle h, f \rangle} = \alpha \overline{\langle g, f \rangle} + \beta \overline{\langle h, f \rangle} = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$$

by (i) and (iv), we can see that condition (ii) is unnecessary.

Let X be an inner product space.

(c) [5p] Let $u, v \in X$. Suppose that $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in X$. Show that $u = v$.

> Rearranging, we have that $0 = \langle u - v, x\rangle$ for all $x \in X$. Since $u - v \in X$, we then have
> $$0 = \langle u - v, u - v\rangle = \|u - v\|^2$$
> which implies that $u - v = 0$.

Now let $X = \mathbb{R}^n$ and define a norm $\|\cdot\|_1 : X \to \mathbb{R}$ by

$$\|x\|_1 := \sum_{j=1}^{n} |x_j|$$

for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. [You do not need to prove that $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$.]

(d) [10p] Show that $\nexists$ an inner product $\langle\cdot,\cdot\rangle_1$ such that

$$\|f\|_1 = \sqrt{\langle f, f\rangle_1}$$

for all $f \in \mathbb{R}^n$.

> The Parallelogram Law (the name of this question was a huge clue) tells us that a norm is associated with an inner product if and only if
> $$\|x + y\|_1^2 + \|x - y\|_1^2 = 2\|x\|_1^2 + 2\|y\|_1^2$$
> for all $x, y \in \mathbb{R}^k$.
> Notice that if $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$ then
> $$\|x + y\|_1^2 + \|x - y\|_1^2 = \big(|1| + |1| + |0| + \ldots + |0|\big)^2 + \big(|1| + |-1| + |0| + \ldots + |0|\big)^2 = 8$$
> but
> $$2\|x\|_1^2 + 2\|y\|_1^2 = 2\big(|1| + |0| + \ldots + |0|\big)^2 + 2\big(|0| + |1| + |0| + \ldots + |0|\big)^2 = 4.$$
> Therefore, there does not exist an inner product associated with $\|\cdot\|_1$.

Soru 5 (Cauchy Sequences and Closed Subspaces). Let $(X, \|\cdot\|_X)$ be a normed space.

(a) [4p] Give the definition of a *Cauchy sequence* in X.

> Let $(f^n)_{n=1}^{\infty}$ be a sequence in X. We say that f^n is a *Cauchy sequence* iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
> $$n, m > N \implies \|f^n - f^m\|_X < \varepsilon.$$

Consider the vector space

$$\ell^2(\mathbb{N}) := \left\{a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \sum_{j=1}^{\infty} |a_j|^2 < \infty\right\}$$

with the inner product

$$\langle f, g\rangle_2 := \sum_{j=1}^{\infty} \overline{f_j} g_j$$

and the norm $\|f\|_2 := \sqrt{\langle f, f\rangle_2}$. Define

$$S := \left\{a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \exists N \in \mathbb{N} \text{ such that } a_n = 0\ \forall n > N\right\}.$$

(b) [7p] Show that S is a subspace of $\ell^2(\mathbb{N})$.

[HINT: The question says "sub*space*", not "sub*set*".]

Clearly, if $b \in S$ then

$$\|b\|_2^2 = \sum_{j=1}^{\infty} |b_j|^2 = \sum_{j=1}^{N} |b_j|^2 < \infty$$

for some $N \in \mathbb{N}$. So $S \subseteq \ell^2(\mathbb{N})$.

Now suppose that $a, b \in S$ and $\lambda \in \mathbb{C}$. Let $N_a, N_b \in \mathbb{N}$ be natural numbers such that $a_n = 0$ for all $n > N_a$, and $b_n = 0$ for all $n > N_b$. Define $N := \max\{N_a, N_b\}$. Then $(a + \lambda b)_n = a_n + \lambda b_n = 0$ for all $n > N$ and hence $a + \lambda b \in S$. Therefore S is a vector space.

Now define a sequence $\{f^n\}_{n=1}^{\infty} \subseteq S$ by

$$f_j^n := \begin{cases} 2^{\frac{1-j}{2}} & j \le n \\ 0 & j > n. \end{cases}$$

For example,

$$f^5 = (1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2^{\frac{3}{2}}}, \frac{1}{4}, 0, 0, 0, 0, 0, \dots).$$

(c) [7p] Show that $\{f^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\ell^2(\mathbb{N})$.

Suppose that $n > m > N$. Then

$$\begin{aligned}
\|f^n - f^m\|_2^2 &= \sum_{j=1}^{\infty} \left|f_j^n - f_j^m\right|^2 \\
&= \sum_{j=1}^{m} \left|f_j^n - f_j^m\right|^2 + \sum_{j=m+1}^{n} \left|f_j^n - f_j^m\right|^2 + \sum_{j=n+1}^{\infty} \left|f_j^n - f_j^m\right|^2 \\
&= 0 + \sum_{j=m+1}^{n} \left|f_j^n\right|^2 + 0 \\
&= \sum_{j=m+1}^{n} 2^{1-j} = 2^{-m} \sum_{j=1}^{n-m} 2^{1-j} \\
&\le 2^{-m} \sum_{j=1}^{\infty} 2^{1-j} = 2^{1-m} \le 2^{-N} \to 0
\end{aligned}$$

as $N \to \infty$. Hence $\{f^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\ell^2(\mathbb{N})$.

(d) [7p] Show that S is not closed.

Define $f = (f_j)_{j=1}^{\infty}$ by

$$f_j := 2^{\frac{1-j}{2}}.$$

Then $\|f\|_2^2 = \sum_{j=1}^{\infty} 2^{1-j} = 2 < \infty$ as above. So $f \in \ell^2(\mathbb{N})$.

It is clear that $f^n \in S$ for each n, but $f \notin S$. We also have that

$$\|f^n - f\|_2^2 = \sum_{j=n+1}^{\infty} 2^{1-j} = 2^{1-n} \to 0$$

as $n \to \infty$. So $\lim_{n\to\infty} f^n = f$. Therefore S is not closed.