2014.11.19 **MAT461 Fonksiyonel Analiz I – Ara Sınavın Çözümleri** N. Course

Soru 1 (Inner Products). Let X be a vector space.

(a) [10p] Give the definition of an *inner product*.

An *inner product* is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that

(i) $\langle \alpha f + \beta g, h \rangle = \overline{\alpha} \langle f, h \rangle + \overline{\beta} \langle g, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$;

(ii) $\langle f, \alpha g + \beta h \rangle = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$ for all $f, g, h \in X$ and for all $\alpha, \beta \in \mathbb{C}$;

(iii) $\langle f, f \rangle > 0$ for all $f \in X$, $f \neq 0$; and

(iv) $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in X$.

Now let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and define $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ as usual. Let $u \in X$ be a unit vector. Let $f \in X$. Define $f_\parallel := \langle u, f \rangle u$ and $f_\perp := f - f_\parallel$.

(b) [10p] Show that u and $f_\perp$ are orthogonal.

Since
$$\langle u, f_\perp \rangle = \langle u, f - \langle u, f \rangle u \rangle = \langle u, f \rangle - \langle u, f \rangle \langle u, u \rangle = \langle u, f \rangle - \langle u, f \rangle = 0$$
we have that u and $f_\perp$ are orthogonal.

Let $\alpha \in \mathbb{C}$. Define $h := \alpha u$.

(c) [10p] Show that $\|f - h\| \geq \|f_\perp\|$.

By Pythagoras, we have that
$$\|f - h\|^2 = \|f - \alpha u\|^2 = \left\|f_\perp + f_\parallel - \alpha u\right\|^2 = \|f_\perp\|^2 + \left\|f_\parallel - \alpha u\right\|^2 \geq \|f_\perp\|^2.$$
Therefore $\|f - h\| \geq \|f_\perp\|$.

Define $U := \{v \in X : v \text{ is parallel to } u\} \subseteq X$.

(d) [10p] Show that $\left\|f - f_\parallel\right\| = \inf_{v \in U} \|f - v\|$.

Clearly $f_\parallel \in U$. Therefore $\left\|f - f_\parallel\right\| \geq \inf_{v \in U} \|f - v\|$.

The "$\leq$" follows immediately from part (c), and we are done.

(e) [10p] Show that if $w \in U$ and $w \neq f_\parallel$, then
$$\|f - w\| > \inf_{v \in U} \|f - v\|.$$

We have
$$\|f - w\|^2 = \|f_\perp\|^2 + \left\|f_\parallel - w\right\|^2 > \|f_\perp\|^2 = \left\|f - f_\parallel\right\|^2 = \inf_{v \in U} \|f - v\|^2$$
since $w \neq f_\parallel$.

Soru 2 (Norms)**.** Let X be a vector space.

(a) [10p] Give the definition of a *norm* on X.

A *norm* is a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies

(i) $\|f\| > 0$ for all $f \in X$, $f \neq 0$;

(ii) $\|\alpha f\| = |\alpha| \, \|f\|$ for all $f \in X$, $\alpha \in \mathbb{C}$ (or $\alpha \in \mathbb{R}$ if X is a real vector space);

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$.

(b) [15p] Show that every norm is continuous.

Let $x_n \in X$ and $x_n \to x \in X$ as $n \to \infty$.

On your homework, you proved that $|\|f\| - \|g\|| \leq \|f - g\|$. Therefore

$$\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| \to 0$$

as $n \to \infty$. So $\lim_{n\to\infty} \|x_n\| = \|x\|$. Hence $\|\cdot\|$ is continuous.

(c) [25p] Now suppose that Y is a finite dimensional complex vector space. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. Then any vector $y \in Y$ can be written as $y = \sum_{j=1}^n \lambda_j e_j$ for unique $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. Define a function $\|\cdot\|_Y : Y \to \mathbb{R}$ by $\|y\|_Y := \left(\sum_{j=1}^n |\lambda_j|^2 \right)^{\frac{1}{2}}$. Show that $\|\cdot\|_Y$ is a norm on Y.

[HINT: You may use the inequality $\sum_{j=1}^k |\alpha_j| \, |\beta_j| \leq \left(\sum_{j=1}^k |\alpha_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^k |\beta_j|^2 \right)^{\frac{1}{2}}$.]

Let $x = \sum \mu_j e_j \in Y$, $y = \sum \lambda_j e_j \in Y$ and $\alpha \in \mathbb{C}$. Then

(a) $\boxed{6}$ If $x \neq 0$, then $\exists k$ such that $\mu_k \neq 0$ and so

$$\|x\|_Y^2 = \sum_j |\mu_j|^2 \geq |\mu_k|^2 > 0.$$

(b) $\boxed{6}$ Clearly

$$\|\alpha x\|_Y^2 = \sum_{j=1}^n |\alpha \mu_j|^2 = |\alpha|^2 \sum_j |\mu_j|^2 = |\alpha|^2 \|x\|_Y^2 .$$

(c) $\boxed{13}$ Finally

$$\begin{aligned}
\|x + y\|_Y^2 &= \sum_j |\mu_j + \lambda_j|^2 \\
&= \sum_j |\mu_j|^2 + \sum_j \overline{\mu_j} \lambda_j + \sum_j \mu_j \overline{\lambda_j} + \sum_j |\lambda_j|^2 \\
&= \|x\|_Y^2 + 2 \sum_j \operatorname{Re}(\mu_j \lambda_j) + \|y\|_Y^2 \\
&\leq \|x\|_Y^2 + 2 \sum_j |\mu_j| \, |\lambda_j| + \|y\|_Y^2 \\
&\leq \|x\|_Y^2 + 2 \left(\sum_j |\mu_j|^2 \right)^{\frac{1}{2}} \left(\sum_j |\lambda_j|^2 \right)^{\frac{1}{2}} + \|y\|_Y^2 \\
&= \|x\|_Y^2 + 2 \|x\|_Y \|y\|_Y + \|y\|_Y^2 \\
&= (\|x\|_Y + \|y\|_Y)^2 .
\end{aligned}$$

Therefore $\|\cdot\|_Y$ is a norm.

Soru 3 (Banach spaces).

(a) [5p] Give the definition of a *Banach space*.

A complete normed space is called a Banach space

Let $I = [a, b] \subseteq \mathbb{R}$ and let

$$C^1(I) := \{f : I \to \mathbb{C} : f \text{ is differentiable and } f' \text{ is continuous}\}.$$

(b) [5p] Show that $C^1(I)$ is a vector space.

Let $f, g \in C^1(I)$ and $\lambda \in \mathbb{C}$. Then $f + \lambda g$ is differentiable and $(f + \lambda g)' = f' + \lambda g'$ is continuous. Therefore $f + \lambda g \in C^1(I)$, and so $C^1(I)$ is a vector space.

Let

$$\|f\|_{\infty,1} := \max_{x \in I} |f(x)| + \max_{x \in I} |f'(x)|.$$

(c) [15p] Show that $\|\cdot\|_{\infty,1}$ is a norm on $C^1(I)$.

(a) If $f \not\equiv 0$, then $\exists x$ such that $f(x) \neq 0$ and so $\|f\|_{\infty,1} > 0$.

(b) $\|\alpha f\|_{\infty,1} = \|\alpha f\|_\infty + \|\alpha f'\|_\infty = |\alpha| \|f\|_\infty + |\alpha| \|f'\|_\infty = |\alpha| \|f\|_{\infty,1}$.

(c) $\|f + g\|_{\infty,1} = \|f + g\|_\infty + \|f' + g'\|_\infty \leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty = \|f\|_{\infty,1} + \|g\|_{\infty,1}$

Therefore $\|\cdot\|_{\infty,1}$ is a norm on $C^1(I)$.

(d) [25p] Show that $(C^1(I), \|\cdot\|_{\infty,1})$ is a Banach space.

[HINT: If f_n is a Cauchy sequence in $C^1(I)$ then f_n and f_n' are Cauchy sequences in $C(I)$. You may assume that $C(I)$ is complete. The Fundamental Theorem of Calculus tells us that $f_n(x) - f_n(a) = \int_a^x f_n'(t)dt$. You may assume that $\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x \lim_{n\to\infty} f_n'(t)dt$.]

Let f_n be a Cauchy sequence in $C^1(I)$. Then f_n and f_n' are Cauchy sequences in $C(I)$ – which is complete. So $f(x) := \lim_{n\to\infty} f_n(x)$ and $g(x) := \lim_{n\to\infty} f_n'(x)$ are continuous. We must prove that f is differentiable and that $f' = g$.

By the Fundamental Theorem of Calculus,

$$f_n(x) - f(x) = \int_a^x f_n'(t)\ dt$$

for all n. So

$$\begin{aligned} f(x) - f(a) &= \lim_{n\to\infty} f_n(x) - f_n(a) \\ &= \lim_{n\to\infty} \int_a^x f_n'(t)\ dt \\ &= \int_a^x \lim_{n\to\infty} f_n'(t)\ dt \\ &= \int_a^x g(t)\ dt. \end{aligned}$$

This proves that f is differentiable and that $f' = g$. Therefore $f \in C^1(I)$ and $\|f_n - f\|_{\infty.1} \to 0$.