2016.01.04 **MAT461 Fonksiyonel Analiz I – Final Sınavın Çözümleri** N. Course

Soru 1 (Operators Defined via Forms). Let X be a Hilbert space.

(a) [4p] Give the definition of a *sesquilinear form* on X.

> A sesquilinear form is a function $s : X \times X \to \mathbb{C}$ such that
>
> (a) $s(\alpha f + \beta g, h) = \bar{\alpha} s(f, h) + \bar{\beta} s(g, h)$; and
>
> (b) $s(f, \alpha g + \beta h) = \alpha s(f, g) + \beta s(f, h)$
>
> for all $f, g \in X$ and for all $\alpha, \beta \in \mathbb{C}$.

(b) [12p] Let $A \in \mathcal{B}(X)$. Show that **there exists** a **unique** operator $A^* \in \mathcal{B}(X)$ such that

$$\langle f, A^* g\rangle = \langle Af, g\rangle$$

for all $f, g \in X$.

> Define $s : X \times X \to \mathbb{C}$ by $s(f, g) = \langle Af, g\rangle$. It is easy to see that s is sesquilinear because A is linear and inner products are sesquilinear.
>
> By Lemma 2.11, it follows that $\exists$ a unique bounded operator A^* such that
>
> $$s(f, g) = \langle f, A^*, g\rangle$$
>
> for all $f, g \in X$ and we are finished.

(c) [1p] What name do we give to A^*?

> A^* is called the *adjoint operator* of A.

(d) [8p] Show that $\|A\| = \|A^*\|$

> By Lemma 2.11
>
> $$\|A^*\| = \sup_{\|f\|=\|g\|=1} |\langle f, A^* g\rangle| = \sup_{\|f\|=\|g\|=1} |\langle Af, g\rangle| = \sup_{\|f\|=\|g\|=1} |\langle g, Af\rangle| = \|A\|.$$

Soru 2 (The Spectral Theorem for Compact Symmetric Operators). Let X be a Hilbert space.

(a) [7p] Suppose that $B : X \to X$ is a bounded operator and suppose that λ is an eigenvalue of B. Show that $|\lambda| \leq \|B\|$.

> A really easy question so that everyone can get some points:
> Suppose that $Bu = \lambda u$ for some unit vector $u \in \mathfrak{D}(B)$. Then by the definition of the operator norm,
>
> $$\|B\| = \sup_{\|f\|=1, f \in \mathfrak{D}(B)} \|Bf\| \geq \|Bu\| = \lambda$$
>
> and we are finished.

(b) [4p] Give the definition of a *symmetrical* operator.

An operator $A : \mathfrak{D}(A) \subseteq X \to X$ is called *symmetrical* iff its domain is dense and if

$$\langle g, Af \rangle = \langle Ag, f \rangle$$

for all $f, g \in \mathfrak{D}(A)$.

Suppose that the linear operator $A : X \to X$ is symmetrical and compact. Suppose that $\alpha_1 \in \mathbb{R}$ is a eigenvalue of A and suppose that $|\alpha_1| = \|A\|$. (We proved in class that it is always possible to find such an eigenvalue.) Let u_1 be a corresponding normalised eigenvector ($\|u_1\| = 1$).

Define

$$X_1 := \{u_1\}^\perp = \{f \in X : \langle u_1, f \rangle = 0\} \subseteq X.$$

Then

$$f \in X_1 \implies \langle u_1, Af \rangle = \langle Au_1, f \rangle = \alpha_1 \langle u_1, f \rangle = 0 \implies Af \in X_1.$$

So we can define a new operator $A_1 : X_1 \to X_1$ by $A_1 f := Af$.

(c) [7p] Show that A_1 is symmetrical.

Let $f, g \in X_1 \subseteq X$. Then

$$\langle f, A_1 g \rangle = \langle f, Ag \rangle = \langle Af, g \rangle = \langle A_1 f, g \rangle$$

since A is symmetrical. Therefore A_1 is also symmetrical.

(d) [7p] Show that A_1 is compact.

Let x_n be a bounded sequence in X_1. But X_1 is a subset of X, so x_n is also a bounded sequence in X. Since A is compact, $\exists$ a subsequence x_{n_k} such that Ax_{n_k} converges. But $A_1 = A|_{X_1}$, so A_1 must also be compact.
(Basically, A_1 is compact because A is compact. Shame on you if you don't get this easy question correct.)

Soru 3 (Orthogonal Complements and Orthogonal Projection). Let X be a Hilbert space. Let M be a closed linear subspace of X.

(a) [3p] Give the definition of a *total* set.

A set $S \subseteq X$ is said to be total in X iff the span of S is dense in X.

(b) [4p] Give the definition of the *orthogonal projection corresponding to* M, P_M.

By the Projection Theorem, $\forall\ f \in X$, $\exists$ unique $f_\parallel \in M$ and $f_perp \in M^\perp$ such that $f = f_\parallel + f_\perp$. The orthogonal projection corresponding to M is the map $P_M : X \to M$ defined by $P_M f := f_\parallel$.

(c) [7p] Calculate $\|P_M\|$.

Clearly $\|P_M\| = 1$.

If $f \in M$, then $P_M f = f_\parallel = f$ which gives us $\|P_M\| \geq 1$. For general $f \in X$, we have that $\|f\|^2 = \left\|f_\parallel\right\|^2 + \|f_\perp\|^2 \geq \left\|f_\parallel\right\|^2 = \|P_M f\|^2$. Thus $\|P_M f\| \leq \|f\|$ for all $f \in X$, which gives us $\|P_M\| \leq 1$.

Let $S \subseteq X$ be a subset of X.

(d) [11p] Show that

$$S^\perp = \{0\} \qquad \Longleftrightarrow \qquad S \text{ is total.}$$

First suppose that S is total. Then

$$S^{\perp\perp} = \overline{\operatorname{span}(S)} = X$$

which implies that $S^\perp = \{0\}$.

Conversely, suppose that $S^\perp = \{0\}$. Then

$$\overline{\operatorname{span}(S)} = S^{\perp\perp} = \{0\}^\perp = X.$$

Hence S is total in X.

Soru 4 (Inner Products). Let X be a vector space.

(a) [5p] Give the definition of an *inner product* on X.

An *inner product* is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ such that

(i) $\langle \alpha f + \beta g, h\rangle = \overline{\alpha}\langle f,h\rangle + \overline{\beta}\langle g,h\rangle$ for all $f,g,h \in X$ and for all $\alpha,\beta \in \mathbb{C}$;

(ii) $\langle f, \alpha g + \beta h\rangle = \alpha\langle f,g\rangle + \beta\langle f,h\rangle$ for all $f,g,h \in X$ and for all $\alpha,\beta \in \mathbb{C}$;

(iii) $\langle f,f\rangle > 0$ for all $f \in X$, $f \neq 0$; and

(iv) $\langle f,g\rangle = \overline{\langle g,f\rangle}$ for all $f,g \in X$.

(b) [5p] Give an example of an inner product space. Prove that your inner product satisfies the definition that you wrote in part (a).

One example would be $\mathbb{R}^2$ with $\langle x,y\rangle = x_1y_1 + x_2y_2$. Proof omitted.

Now let X be a Hilbert space. The *Parallelogram Law* tell us that

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

for all $f,g \in X$. The *Generalised Parallelogram Law* states that

$$\left\|\sum_{j=1}^{n} x_j\right\|^2 + \sum_{1\le j<k\le n} \|x_j - x_k\|^2 = n\sum_{j=1}^{n}\|x_j\|^2 \tag{1}$$

for all $\{x_1, x_2, \dots, x_n\} \subseteq X$. Note that the case $n = 2$ is the same as the Parallelogram Law.

(c) [15p] Prove the Generalised Parallelogram Law.

Let $x_1, x_2, \ldots, x_n$ be any vectors in X. First we calculate that

$$\begin{aligned}\left\|\sum_{j=1}^n x_j\right\|^2 &= \left\langle \sum_{j=1}^n x_j, \sum_{k=1}^n x_k \right\rangle = \sum_{j=1}^n \sum_{k=1}^n \langle x_j, x_k\rangle \\ &= \sum_j \|x_j\|^2 + \sum_{j\neq k} \langle x_j, x_k\rangle \\ &= \sum_j \|x_j\|^2 + \sum_{j<k} \left(\langle x_j, x_k\rangle + \langle x_k, x_j\rangle\right).\end{aligned}$$

Moreover

$$\begin{aligned}\sum_{1\le j<k\le n} \|x_j - x_k\|^2 &= \sum_{j<k} \langle x_j - x_k, x_j - x_k\rangle \\ &= \sum_{j<k} \left(\|x_j\|^2 - \langle x_j, x_k\rangle - \langle x_k, x_j\rangle + \|x_k\|^2\right) \\ &= (n-1)\sum_j \|x_j\|^2 - \sum_{j<k} \left(\langle x_j, x_k\rangle + \langle x_k, x_j\rangle\right)\end{aligned}$$

Taking the sum of these two equations then gives (1).

Soru 5 (Compact Operators). Let X be a normed space.

(a) [3p] Give the definition of a *compact set*.

A set $S \subseteq X$ is called *compact* iff every open cover has a finite subcover.

(b) [5p] Give the definition of a *compact operator*.

An operator K is called a *compact* operator iff, for all bounded sequences $(x_n) \subseteq X$ there exists a subsequence (x_{n_j}) such that $(Kx_{n_j}) \subseteq Y$ is convergent.

(c) [5p] Give an example of a compact operator. Prove that your operator is compact.

There are many many many simple examples you could give.

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Af = 7f$. Every operator on a finite dimensional vector space is bounded, and every bounded operator is compact. So this operator is clearly compact.

Let $K : X \to X$ be a compact operator. Let $\bar{X}$ denote the completion of X. By the B.L.T. Theorem, $\exists$ a unique continuous extension of K to $\bar{X}$. Let $\bar{K} : \bar{X} \to \bar{X}$ denote this extension.

(d) [12p] Show that $\bar{K}$ is a compact operator.

Let $f_n \in \bar{X}$ be a bounded sequence. We need to show that Af_n has a convergent subsequence.

For each n , choose a sequence $(g_{n,j})_{j=1}^\infty \subseteq X$ such that $g_{n,j} \to f_n \in \bar{X}$. We can always do this because X is dense in $\bar{X}$. The sequence $(g_{n,n})$ must be bounded because f_n is bounded. Since K is compact, $\exists$ a subsequence such that $Kg_{n_j,n_j} \to g \in \bar{X}$.

But then

$$\left\|\bar{K} f_{n_j} - g\right\| \le \|K\| \left\|f_{n_j} - g_{n_j,n_j}\right\| + \left\|K g_{n_j,n_j} - g\right\| \to 0.$$

So $\bar{K} f_{n_j} \to g$.

Therefore $\bar{K}$ is compact.