2015.11.10 **MAT461 Fonksiyonel Analiz I – Ara Sınavın Çözümleri** N. Course

Soru 1 (Norms). Let X be a vector space.

(a) [10p] Give the definition of a *norm* on X.

A *norm* is a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies

(i) $\|f\| > 0$ for all $f \in X$, $f \neq 0$;

(ii) $\|\alpha f\| = |\alpha| \, \|f\|$ for all $f \in X$, $\alpha \in \mathbb{C}$ (or $\alpha \in \mathbb{R}$ if X is a real vector space);

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$.

Let $p \in (0,1)$. Define

$$\ell^p(\mathbb{N}) := \{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_p < \infty\}$$

where

$$\|a\|_p = \left(\sum_{j=1}^\infty |a_j|^p \right)^{\frac{1}{p}} .$$

(b) [14p] Show that $\|\cdot\|_p$ does <u>not</u> satisfy the triangle inequality.
[HINT: Don't forget that $0 < p < 1$.]

Let $a := (1, 0, 0, 0, \ldots)$ and $b := (0, 1, 0, 0, \ldots)$. Then clearly $\|a\|_p = 1 = \|b\|_p$. However

$$\|a + b\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} > 2 = \|a\|_p + \|b\|_p .$$

(c) [13p] Show that $\|\cdot\|_p$ satisfies the other two conditions in the definition of a norm.

Clearly if $f \neq 0$ then $f_j \neq 0$ for some k. Hence $\|f\|_p = \left(\sum_{j=1}^\infty |f_j|^p \right)^{\frac{1}{p}} \geq (|f_k|^p)^{\frac{1}{p}} = |f_k| > 0$. Moreover, we have that $\|\alpha f\|_p^p = \sum_{j=1}^\infty |\alpha f_j|^p = |\alpha|^p \sum_{j=1}^\infty |f_j|^p = |\alpha|^p \|f\|_p^p$.

(d) [13p] Show that

$$\|a + b\|_p \leq 2^{\frac{1}{p-1}} \left(\|a\|_p + \|b\|_p \right)$$

for all $a, b \in \ell^p(\mathbb{N})$.

[HINT: $\alpha + \beta \leq (\alpha^p + \beta^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p-1}} (\alpha + \beta)$ for all $\alpha, \beta \geq 0$ and $0 < p < 1$.]

By the hint,

$$\|a + b\|_p^p = \sum_j |a_j + b_j|^p \leq \sum_j (|a_j| + |b_j|)^p \leq \sum_j |a_j|^p + |b_j|^p$$
$$= \|a\|_p^p + \|b\|_p^p \leq 2^{\frac{p}{p-1}} \left(\|a\|_p + \|b\|_p \right)^p .$$

You have proved that $\|\cdot\|_p$ is a *quasinorm* on $\ell^p(\mathbb{N})$.

Soru 2 (Separable Hilbert Spaces).

(a) [5p] Give the definition of a *Hilbert space.*

A complete inner product space is called a Hilbert space.

(b) [5p] Give definition of a *separable* space.

A space is called separable iff it contains a countable dense subset.

(c) [15p] Show that $\mathbb{C}^7$, with the function $\langle f, g\rangle := \sum_{j=1}^{7} \bar{f}_j g_j$, is a Hilbert space.

Since

(a) $\langle \alpha f + g, h\rangle = \sum \overline{(\alpha f_j + g_j)} h_j = \sum \bar{\alpha} \bar{f}_j h_j + \sum \bar{g}_j h_j = \bar{\alpha} \langle f, h\rangle + \langle g, h\rangle$;

(b) $\langle f, f\rangle = \sum |f_j|^2 > 0$ if $f \neq 0$; and

(c) $\overline{\langle f, g\rangle} = \overline{\sum \bar{f}_j g_j} = \sum \bar{\bar{f}}_j \bar{g}_j = \sum \bar{g}_j f_j = \langle g, f\rangle$,

$\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^7$.

Clearly $\mathbb{C}^7$ is complete because $\mathbb{C}$ is.

Now consider the Hilbert space

$$\ell^2(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \sum_{j=1}^{\infty} |a_j|^2 < \infty \right\}$$

with the inner product

$$\langle a, b\rangle_2 := \sum_{j=1}^{\infty} \bar{a}_j b_j.$$

(d) [25p] Show that $(\ell^2(\mathbb{N}), \langle \cdot, \cdot \rangle_2)$ is separable.

[HINT: You might like to consider the set $A = \{a \in \ell^2(\mathbb{N}) : \mathrm{Re}(a_j), \mathrm{Im}(a_j) \in \mathbb{Q}$, only finitely many of the a_j are non-zero$\}$.]

Let

$$A_N := \left\{ a \in \ell^2(\mathbb{N}) : \mathrm{Re}(a_j), \mathrm{Im}(a_j) \in \mathbb{Q} \text{ and } a_j = 0 \ \forall j > N \right\}.$$

Clearly A_N is countable because $\mathbb{Q}$ is. Hence $A := \bigcup_N A_N$ is countable.

Let $\varepsilon > 0$ and let $a \in \ell^2(\mathbb{N}) \cap \mathbb{Q}^{\mathbb{N}}$. Then there exists N such that $\sum_{j=N+1}^{\infty} |a_j|^2 < \varepsilon^2$. Define

$$a_j^N = \begin{cases} a_j & 1 \leq j \leq N \\ 0 & j > N. \end{cases}$$

Then $\left\| a - a^N \right\|_2 = \sqrt{\sum_{j=N+1}^{\infty} |a_j|^2} < \varepsilon$ and $a^N \in A$. So A is dense in $\ell^2(\mathbb{N}) \cap \mathbb{Q}^{\mathbb{N}}$. Since $\ell^2(\mathbb{N}) \cap \mathbb{Q}^{\mathbb{N}}$ is dense in $\ell^2(\mathbb{N})$, we have that $\ell^2(\mathbb{N})$ is separable.

Soru 3 (Bounded Linear Operators). Let X and Y be normed spaces.

(a) [10p] Give the definition of the *Operator Norm.*

The operator norm of $A : \mathfrak{D}(A) \subseteq X \to Y$ is defined to be

$$\|A\| := \sup_{\substack{f \in \mathfrak{D}(A) \\ \|f\|_X = 1}} \|Af\|_Y .$$

Now let $A_n, B_n, A, B \in \mathcal{B}(X)$. Suppose that $A_n \to A$ and $B_n \to B$.

(b) [15p] Show that $A_n B_n \to AB$.
[HINT: You may assume without proof that $\|AB\| \leq \|A\| \|B\|$ for all $A, B \in \mathcal{B}(X)$.]

Since $A_n \to A$, we know that $\|A_n - A\| \to 0$. Similarly $\|B_n - B\| \to 0$. For sufficiently large n, we can assume that $\|B_n\| \leq 1 + \|B\| < \infty$. Clearly $\|A\| < \infty$.

Then we have that

$$\begin{aligned}\|A_n B_n - AB\| &= \|A_n B_n - AB_n + AB_n - AB\| \leq \|A_n - A\| \|B_n\| + \|A\| \|B_n - B\| \\ &\leq \|A_n - A\| (1 + \|B\|) + \|A\| \|B_n - B\| \to 0.\end{aligned}$$

(c) [25p] Let $T \in \mathcal{B}(X)$ be a bijection. Show that

$$\left\|T^{-1}\right\|^{-1} = \inf_{\substack{f \in X \\ \|f\|_X = 1}} \|Tf\|_X .$$

As notation, let $g = Tf$. Then $f = T^{-1}g$. Because T is a bijection, we know that $f = 0$ if and only if $g = 0$.

Now,

$$\begin{aligned}\left\|T^{-1}\right\|^{-1} &= \frac{1}{\|T^{-1}\|} = \frac{1}{\sup_{\|g\|=1} \|T^{-1}g\|} \\ &= \inf_{\|g\|=1} \frac{1}{\|T^{-1}g\|} = \inf_{g \neq 0} \frac{\|g\|}{\|T^{-1}g\|} \\ &= \inf_{f \neq 0} \frac{\|Tf\|}{\|f\|} = \inf_{\|f\|=1} \|Tf\|\end{aligned}$$

and we are finished.