2017.01.03 **MAT461 Fonksiyonel Analiz I – Final Sınavın Çözümleri** N. Course

Soru 1 (Bounded Operators). [25p] Please write two pages about *bounded operators*.

There are many possible answers. I am hoping to see the definition, some interesting examples that we didn't cover in class, and statements (with or without proofs) of the basic lemmata and theorems.
Marks will be given generously for this question.

Soru 2 (Equicontinuous sets of functions).

(a) [5p] Give the definition of a *equicontinuous* set of functions $F = \{f_j : X \to Y : j \in J\}$.

F is equicontinuous iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that

$$\|x - y\| < \delta \quad \Longrightarrow \quad \|f(x) - f(y)\| < \varepsilon$$

for all $f \in F$.

(b) [8p] Show that every bounded sequence in $\left(C^1([a,b]), \|\cdot\|_{\infty,1}\right)$ is equicontinuous.

Let f_n be a bounded sequence in C^1. Suppose that $\|f_n\|_{\infty,1} \leq M$ for all n. Since

$$|x-y| < \delta \quad \Longrightarrow |f(x) - f(y)| \leq \max_{z\in[a,b]} |f'(z)|\,|x-y| \leq \|f\|_{\infty,1}\,|x-y| \leq M\delta,$$

we have that $\{f_n\}$ is equicontinuous.

(c) [12p] Show that the operator $\frac{d}{dx} : \left(C^1([a,b]), \|\cdot\|_{\infty,1}\right) \to \left(C([a,b]), \|\cdot\|_{\infty}\right)$ is compact.
[HINT: Arzelà-Ascoli.]

Let f_n be a bounded sequence in C^1. We must show that f_n' has a convergent subsequence.

By part (b), we know that f_n is equicontinous. It follows by Arzelà-Ascoli that f_n has a uniformly convergent subsequence, f_{n_k}. By choice of norms, f_{n_k}' is also convergent, and we are finished.

Soru 3 (Compact Operators).

(a) [5p] Give the definition of a compact operator.

An operator K is called a *compact* operator iff, for all bounded sequences $(x_n) \subseteq X$ there exists a subsequence (x_{n_j}) such that $(Kx_{n_j}) \subseteq Y$ is convergent.

(b) [7p] Give an example of a compact operator. Justify your answer.

There are many possible answers. I'm hoping for something original.

(c) [6p] Show that every compact operator is bounded.

Suppose that $A \in \mathcal{K}(X,Y)$ is not bounded. Then $\forall\ n \in \mathbb{N}$, $\exists$ a unit vector u_n such that $\|Au_n\| \geq n$. Since the sequence $\{u_n\}$ is bounded and since A is compact, $\exists$ a convergent subsequence $\{Au_{n_j}\}$. But this contradicts $\|Au_n\| \geq n$. So A must be bounded.

(d) [7p] Give an example of an operator which is <u>not</u> compact. Justify your answer.

Since every compact operator is bounded, it suffices to give an unbounded operator.

Soru 4 (Self-Adjoint Linear Operators). Let X be a Hilbert space.

(a) [2p] Give the definition of the *adjoint* of an operator $A : X \to X$.

The adjoint of $A : X \to X$ is the unique bounded linear operator A^* such that

$$\langle f, Ag\rangle = \langle A^* f, g\rangle$$

for all $f, g \in X$.

(b) [3p] Give the definition of a *self-adjoint* operator.

An operator $A : X \to X$ is called self-adjoint if and only if $A = A^*$.

Now let $T \in \mathcal{B}(X)$ be any bounded linear operator. Define $R : X \to X$ and $S : X \to X$ by

$$R := \frac{1}{2}(T + T^*) \qquad \text{and} \qquad S := \frac{1}{2i}(T - T^*).$$

Note that T is a linear combination of R and S ($T = R + iS$).

(c) [4p] Show that $\langle Rf, f\rangle \in \mathbb{R}$ for all $f \in X$

[HINT: Recall that $z + \bar{z} = 2\,\mathrm{Re}(z)$ for all $z \in \mathbb{C}$.]

We have that

$$\begin{aligned} 2\langle Rf, f\rangle &= \langle (T + T^*)f, f\rangle = \langle Tf, f\rangle + \langle T^* f, f\rangle = \langle Tf, f\rangle + \langle f, Tf\rangle \\ &= \langle Tf, f\rangle + \overline{\langle Tf, f\rangle} = 2\,\mathrm{Re}\,\langle Tf, f\rangle \in \mathbb{R} \end{aligned}$$

for all $f \in X$.

(d) [4p] Show that $\langle Sf, f\rangle \in \mathbb{R}$ for all $f \in X$.

Similarly to the previous part, we have that

$$\begin{aligned} -2i\langle Sf, f\rangle = \langle 2iSf, f\rangle &= \langle (T - T^*)f, f\rangle = \langle Tf, f\rangle - \langle T^* f, f\rangle \\ &= \langle Tf, f\rangle - \langle f, Tf\rangle = \langle Tf, f\rangle - \overline{\langle Tf, f\rangle} = i\,\mathrm{Im}\,\langle Tf, f\rangle \end{aligned}$$

and hence that $\langle Sf, f\rangle \in \mathbb{R}$ for all $f \in X$.

$$R := \frac{1}{2}(T + T^*) \qquad\qquad S := \frac{1}{2i}(T - T^*)$$

(e) [6p] Show that $R : X \to X$ is self-adjoint.

And an easy 12 points to finish this question: Clearly

$$\begin{aligned} \langle Rf, g\rangle &= \left\langle \frac{1}{2}(T + T^*)f, g \right\rangle = \frac{1}{2}\Big(\langle Tf, g\rangle + \langle T^* f, g\rangle \Big) = \frac{1}{2}\Big(\langle f, T^* g\rangle + \langle f, Tg\rangle \Big) \\ &= \langle f, Rg\rangle = \langle R^* f, g\rangle \end{aligned}$$

for all $f, g \in X$. Therefore $R = R^*$ and hence R is self-adjoint.

(f) [6p] Show that $S : X \to X$ is self-adjoint.

Similarly

$$\langle Sf, g\rangle = \left\langle \frac{1}{2i}(T - T^*)f, g \right\rangle = -\frac{1}{2i}\Big(\langle Tf, g\rangle - \langle T^* f, g\rangle \Big) = -\frac{1}{2i}\Big(\langle f, T^* g\rangle - \langle f, Tg\rangle \Big)$$
$$= \frac{1}{2i}\Big(\langle f, Tg\rangle - \langle f, T^* g\rangle \Big) = \left\langle f, \frac{1}{2i}(T - T^*)g \right\rangle = \langle f, Sg\rangle = \langle S^* f, g\rangle$$

for all $f, g \in X$. Therefore $S = S^*$ and hence S is self-adjoint.

Soru 5 (Bounded Operators). Let $f_j \in \mathbb{C}$ for all j. Suppose that . Suppose that

- $f(z) := \sum_{j=0}^{\infty} f_j z^j$ $(z \in \mathbb{C})$ is a power series with radius of convergence $R > 0$ (i.e. the power series converges absolutely if $|z| < R$);
- X is a Banach space;
- $A \in \mathcal{B}(X)$ is a bounded operator; and
- $\limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} < R$.

[25p] Show that

$$f(A) := \sum_{j=0}^{\infty} f_j A^j$$

exists and is a bounded operator $f(A) : X \to X$.

Clearly

$$\sum_{j=0}^{\infty} \left\| f_j A^j x \right\| \leq \sum_{j=0}^{\infty} |f_j| \left\| A^j \right\| \|x\| = \|x\| \sum_{j=0}^{\infty} |f_j| \left\| A^j \right\| = \|x\| \sum_{j=0}^{\infty} \left| f_j \left(\left\| A^j \right\|^{\frac{1}{j}} \right)^j \right| \tag{1}$$

for all $x \in X$. Since $\left\| A^j \right\|^{\frac{1}{j}}$ is a real number in $[0, R) \subseteq (-R, R)$ for sufficiently large j, this series converges by the definition of the radius of convergence. It follows that $f(A)$ exists.

Moreover, if follows from (1) that

$$\|f(A)\| \leq \sum_{j=0}^{\infty} \left| f_j \left(\left\| A^j \right\|^{\frac{1}{j}} \right)^j \right| < \infty.$$

Therefore $f(A)$ is a bounded operator.