2016.11.15 **MAT461 Fonksiyonel Analiz I – Ara Sınavın Çözümleri** N. Course

Soru 1 (Quotients of Banach Spaces). When we proved Lemma 1.31 in class, we skipped the easier parts – I just wrote "you prove". Now you will fill in the gaps to complete the proof.

Definition. Let X be a Banach space and let $M \subseteq X$ be a closed subspace. The *quotient space* X/M is the set of all equivalence classes

$$[x] = x + M$$

with respect to the equivalence relation

$$x \sim \tilde{x} \iff x - \tilde{x} \in M.$$

We define $[x] + [y] = [x + y]$ and $\alpha[x] = [\alpha x]$.

Lemma 1.31. *Let X be a Banach space and let $M \subseteq X$ be a closed subspace. Then X/M together with*

$$\|[x]\| := \inf_{z \in M} \|x + z\|_X \tag{1}$$

is a Banach space.

Proof of Lemma 1.31 We need to prove that (i) X/M is a vector space; (ii) $\|\cdot\|$ is a norm on X/M; and (iii) X/M (with this norm) is complete.

(a) [17p] Show that the definitions $[x] + [y] = [x + y]$ and $\alpha[x] = [\alpha x]$ are well defined. That is, show that these definitions are independent of the choice of representative of the equivalence class.

A very easy one to start with: Suppose that $x_1, x_2 \in [x]$ and $y_1, y_2 \in [y]$.
Then $x_1 \sim x_2$ and $y_1 \sim y_2$. So $x_1 - x_2, y_1 - y_2 \in M$. Because M is a closed subspace, it follows that $(x_1 + y_1) - (x_2 + y_2) \in M$. Therefore $x_1 + y_1 \sim x_2 + y_2$ and hence that $[x_1 + y_1] = [x_2 + y_2]$.
Similarly $\alpha x_1 - \alpha x_2 = \alpha(x_1 - x_2) \in M$ which implies that $\alpha x_1 \sim \alpha x_2$ and hence that $[\alpha x_1] = [\alpha x_2]$.

This proves that X/M is a vector space. Next we prove that $\|\cdot\|$ is a norm on X/M.

First suppose that $\|[x]\| = 0$. Then $\exists$ a sequence $z_j \in M$ such that $z_j \to x$. But M is closed. So we must have that $x \in M$ also. Therefore $[x] = [0]$.

To show that $\|\alpha[x]\| = |\alpha| \, \|[x]\|$, we calculate that

$$\|\alpha[x]\| = \|[\alpha x]\| = \inf_{z \in M} \|\alpha x + z\|_X = \inf_{w \in M} \|\alpha x + \alpha w\|_X = |\alpha| \inf_{w \in M} \|x + w\|_X = |\alpha| \, \|[x]\| \,.$$

Next we must prove that $\|\cdot\|$ satisfies the triangle inequality.

(b) [17p] Prove that $\|[x] + [y]\| \leq \|[x]\| + \|[y]\|$ for all $[x], [y] \in X/M$.

Clearly

$$\begin{aligned} \|[x] + [y]\| = \|[x + y]\| = \inf_{z \in M} \|x + y + z\|_X &= \inf_{z_1, z_2 \in M} \|x + y + z_1 + z_2\|_X \\ &\leq \inf_{z_1, z_2 \in M} \|x + z_1\|_X + \|y + z_2\|_X \\ &= \inf_{z_1, z_2 \in M} \|x + z_1\|_X + \inf_{z_1, z_2 \in M} \|y + z_2\|_X \\ &= \inf_{z_1 \in M} \|x + z_1\|_X + \inf_{z_2 \in M} \|y + z_2\|_X = \|[x]\| + \|[y]\| \,. \end{aligned}$$

Therefore $\|\cdot\|$ is a norm on X/M. Finally we must prove that X/M is complete.

Let $[x_n]$ be a Cauchy sequence. Since it is sufficient to show that a subsequence is convergent, we can assume without loss of generality that

$$\|[x_{n+1}] - [x_n]\| < 2^{-n}.$$

By (1), we can choose the representatives x_n such that

$$\|x_{n+1} - x_n\|_X \leq 2^{-n}.$$

Then x_n is a Cauchy sequence in X. Since X is a Banach space, there exists a limit $x := \lim_{n\to\infty} x_n$ in X. All that remains is to prove that $[x_n] \to [x]$.

(c) [16p] Show that $\|[x_n] - [x]\| \to 0$.

It is easy to see that

$$\begin{aligned}\|[x_n] - [x]\| = \|[x_n - x]\| &= \inf_{z\in M} \|x_n - x + z\|_X \leq \inf_{z\in M} \|x_n - x\|_X + \|z\|_X \\ &= \|x_n - x\|_X + \inf_{z\in M} \|z\|_X = \|x_n - x\|_X + \|[0]\| = \|x_n - x\|_X + 0 \to 0.\end{aligned}$$

Therefore X/M is a Banach space. □

Soru 2 (Bounded and Unbounded Linear Operators).

(a) [5p] Give the definition of the *Operator Norm*.

The operator norm of $A : \mathfrak{D}(A) \subseteq X \to Y$ is defined to be

$$\|A\| := \sup_{\substack{f\in\mathfrak{D}(A)\\ \|f\|_X=1}} \|Af\|_Y .$$

(b) [5p] Give the definition of a *bounded* operator.

An operator $A : (A) \to Y$ is bounded if and only if $\|A\| < \infty$.

(c) [5p] Give the definition of the *kernal* of an operator.

$\mathrm{Ker}(A) = \{f \in (A) : Af = 0\} \subseteq X$.

Let X be a normed space. Let $l : X \to \mathbb{C}$ be a linear map.

(d) [10p] Show that if l is continuous, then the kernal of l is closed.

First suppose that l is continuous. If x_n is a sequence in $\mathrm{Ker}(l)$ and if $x_n \to x$, then $0 = l(x_n) \to l(x)$ and hence $x \in \mathrm{Ker}(l)$ also. Hence $\mathrm{Ker}(l)$ is closed.

(e) [15p] Show that if l is not continuous, then there exists a sequence of unit vectors $u_n \in X$ ($\|u_n\| = 1$) such that $|l(u_n)| \to \infty$ and there exists a vector $y \in X$ such that $l(y) = 1$.

We showed in class that a linear operator is continuous if and only if it is bounded. If l is not continuous, then l is not bounded. It follows immediately from the definition of the operator norm that if $\sup_{\|f\|_X=1} |l(f)| = \infty$, then there must be a sequence of unit vectors u_n such that $|l(u_n)| \to \infty$.

Finally, just define $y := \frac{u_1}{l(u_1)}$. Because l is linear, $l(y) = l\left(\frac{u_1}{l(u_1)}\right) = \frac{l(u_1)}{l(u_1)} = 1$.

(f) [10p] Show that if the kernal of l is closed, then l is continuous.
[HINT: Consider the sequence $x_n = y - \frac{u_n}{l(u_n)}$.]

Suppose that l is not continuous. Let u_n and y be as in part (d). Define a new sequence by $x_n = y - \frac{u_n}{l(u_n)}$. Clearly $l(x_n) = l(y) - \frac{l(u_n)}{l(u_n)} = 1 - 1 = 0$. So $x_n \in \operatorname{Ker}(l)$ for all n. Moreover, since

$$\|x_n - y\| = \left\| \frac{u_n}{l(u_n)} \right\| = \frac{\|u_n\|}{l(u_n)} = \frac{1}{l(u_n)} \to 0,$$

we have that $x_n \to y \notin \operatorname{Ker}(l)$. Therefore $\operatorname{Ker}(l)$ is not closed.

Soru 3 (Norms). Let X be a vector space.

(a) [10p] Give the definition of a *norm* on X.

A *norm* is a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies

(i) $\|f\| > 0$ for all $f \in X$, $f \neq 0$;

(ii) $\|\alpha f\| = |\alpha| \, \|f\|$ for all $f \in X$, $\alpha \in \mathbb{C}$ (or $\alpha \in \mathbb{R}$ if X is a real vector space);

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$.

Consider the following three conditions:

(a) If $\|x + y\| = \|x\| + \|y\|$, then either $x = 0$ or $y = 0$ or $\exists\, \alpha > 0$ such that $y = \alpha x$.
(b) If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|\lambda x + (1-\lambda)y\| < 1$ for all $0 < \lambda < 1$.
(c) If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\frac{1}{2}\|x + y\| < 1$.

A norm which satisfies one of these conditions is called a *strictly convex* norm.

(b) [25p] Show that (i) $\Longrightarrow$ (ii).

(c) [15p] Show that (ii) $\Longrightarrow$ (iii).

[For bonus points, show that (iii) $\Longrightarrow$ (i). This is harder.]

(i) $\Longrightarrow$ (ii): First suppose that $\|x\| = \|y\| = 1$, that $x \neq y$, that $0 < \lambda < 1$, and that (i) is true. Clearly

$$\begin{aligned}\|\lambda x + (1-\lambda)y\| &\leq \lambda\|x\| + (1-\lambda)\|y\| = \lambda + (1-\lambda) = 1 = \lambda\|x\| + (1-\lambda)\|y\| \\ &= \|\lambda x\| + \|(1-\lambda)y\|.\end{aligned}$$

We need to show that the "$\leq$" in this inequality can not be an "$=$".

Using proof by contradiction, we suppose that $\|\lambda x + (1-\lambda)y\| = \|\lambda x\| + \|(1-\lambda)y\|$. Since $\lambda x \neq 0 \neq (1-\lambda)y$, part (i) tells us that we must have $(1-\lambda)y = \alpha\lambda x$ for some $\alpha = \alpha(\lambda) > 0$. It follows that

$$1 = \|y\| = \left\| \frac{\alpha\lambda}{1-\lambda} x \right\| = \frac{\alpha\lambda}{1-\lambda}\|x\| = \frac{\alpha\lambda}{1-\lambda}.$$

Substituting into $(1-\lambda)y = \alpha\lambda x$ gives $y = x$ which is our contradiction.

(ii) $\Longrightarrow$ (iii): Suppose that $\|x\| = \|y\| = 1$, that $x \neq y$ and that (ii) is true. Choosing $\lambda = \frac{1}{2}$ we see that

$$1 > \|\lambda x + (1-\lambda)y\| = \left\| \frac{1}{2}x + \frac{1}{2}y \right\| = \frac{1}{2}\|x + y\|$$

as required.