2016–17 **MAT462 Fonksiyonel Analiz II – Ödev 2** N. Course

SON TESLİM TARİHİ: Pazartesi 20 Mart 2017 saat 17:00'e kadar.

**Egzersiz 4 (The Graph of an Operator).** [25p] Let $X$ and $Y$ be Banach spaces and let $A : X \to Y$ be a linear map (defined on all of $X$). Show that $\Gamma(A)$ is a subspace of $X \oplus Y$.

**Egzersiz 5 (Proof of the Closed Graph Theorem).** [25p] Let $X$ and $Y$ be Banach spaces and let $A : X \to Y$ be a linear map (defined on all of $X$). Show that

$$A \text{ is bounded} \implies A \text{ has closed graph.}$$

[HINT: Start with a Cauchy sequence $(x_n, Ax_n)$ in $\Gamma(A)$. Can you prove that $\lim_{n\to\infty}(x_n, Ax_n)$ exists and is in $\Gamma(A)$? What does this tell us?]

**Egzersiz 6 (Dual Space).** Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider the Banach spaces $\ell^p(\mathbb{N})$, $\ell^q(\mathbb{N})$ and $\ell^p(\mathbb{N})^*$, where

$$\ell^p(\mathbb{N}) := \Big\{ a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_p := \Big( \sum_{j=1}^\infty |a_j|^p \Big)^{\frac{1}{p}} < \infty \Big\}.$$

Let $b = (b_j)_{j=1}^\infty \in \ell^q(\mathbb{N})$. Define

$$a_j = \begin{cases} \frac{|b_j|^q}{b_j} & \text{if } b_j \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) [10p] Show that $a = (a_j)_{j=1}^\infty \in \ell^p(\mathbb{N})$.
[HINT: $\frac{1}{p} + \frac{1}{q} = 1 \iff \frac{q+p}{pq} = 1 \iff \ldots$. Show first that $\|a\|_p^p = \|b\|_q^q$.]

(b) [10p] Show that $\|b\|_q^{q-1} = \|a\|_p$.

For each $y \in \ell^q(\mathbb{N})$, define $l_y : \ell^p(\mathbb{N}) \to \mathbb{C}$ by

$$l_y(x) = \sum_{j=1}^\infty y_j x_j.$$

(c) [10p] Use the Hölder Inequality to show that $\|l_y\| \leq \|y\|_q$ for all $y \in \ell^q(\mathbb{N})$.

(d) [15p] Show that $\|l_y\| = \|y\|_q$ for all $y \in \ell^q(\mathbb{N})$.
[HINT: Choose $x \in \ell^p(\mathbb{N})$ such that $x_j y_j = |y_j|^q$. Why can we always do this? Use part (b).]

(e) [5p] Show that $l_y \in \ell^p(\mathbb{N})^*$ for all $y \in \ell^q(\mathbb{N})$.

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*Ödev 1'in çözümleri*

1. That $U$ is closed $\iff$ $X \setminus U$ is open, is trivial. Note that $U$ is closed and nowhere dense if and only if $U^\circ = \emptyset$. Moreover, $X \setminus U$ is dense if and only if $\overline{X \setminus U} = X$. The result then follows from the identity $X \setminus U^\circ = \overline{X \setminus U}$.

2. Since $A^{-1}$ is continuous $\iff$ $A$ is open, the result follows by the Open Mapping Theorem.

3. Suppose that $\mathfrak{D}(A) \subseteq X$ and $\mathrm{Ran}(A) \subseteq Y$. Since $A$ is closed, its graph $\Gamma(A) = \{(x, Ax) : x \in \mathfrak{D}(A)\} \subseteq X \oplus Y$ is a closed set. But $\Gamma(A^{-1}) = \{(y, A^{-1}y) : y \in \mathrm{Ran}(A)\} = \{(Ax, x) : x \in \mathfrak{D}(A)\} \subseteq Y \oplus X$ is clearly isomorphic to $\Gamma(A)$ so must be closed also. Therefore $A^{-1}$ is closed.