2016–17 **MAT462 Fonksiyonel Analiz II – Ödev 3** N. Course

SON TESLİM TARİHİ: Pazartesi 17 Nisan 2017 saat 17:00'e kadar.

Egzersiz 7 (Proof of Corollary 4.13). Let X be a normed vector space and let $Y \subseteq X$ be a subspace. Define

$$Q := \{ f \in X^* : f(y) = 0 \ \ \forall y \in Y \} \subset X^*.$$

(a) [25p] Show that

$$x_0 \in \overline{Y} \qquad \Longrightarrow \qquad l(x_0) = 0 \quad \forall l \in Q.$$

(b) [25p] Show that

$$x_0 \in \overline{Y} \qquad \Longleftarrow \qquad l(x_0) = 0 \quad \forall l \in Q.$$

Egzersiz 8 (Weak Convergence). Define

$$\delta_j^n = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j. \end{cases}$$

For example, δ^5 is the sequence $(0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)$

(a) [10p] Show that $\delta^n \in \ell^p(\mathbb{N})$ for all $n \in \mathbb{N}$ and for all $1 \leq p \leq \infty$.
[HINT: Don't forget $p = \infty$.]

(b) [10p] Let $1 \leq p \leq \infty$. Show that $\delta^n \not\to 0$ in $\ell^p(\mathbb{N})$.

(c) [15p] Let $1 < p < \infty$. Show that $\delta^n \rightharpoonup 0$ in $\ell^p(\mathbb{N})$.

(d) [15p] Show that δ^n is not weakly convergent in $\ell^1(\mathbb{N})$.
[HINT: First find 2 functionals $l, \tilde{l} \in \ell^1(\mathbb{N})^*$ such that $\lim_{n\to\infty} l(\delta^n) \neq \lim_{n\to\infty} \tilde{l}(\delta^n)$. What does this tell us?]

Ödev 2'nin çözümleri

4. Let (f, Af), $(g, Ag) \in \Gamma(A)$. Let $\lambda \in \mathbb{C}$. Then $f, g \in X$ which is a vector space, so $f + \lambda g \in X$. Moreover, A is linear so $A(f + \lambda g) = Af + \lambda Ag$. Therefore $(f, Af) + \lambda(g, Ag) = (f + \lambda g, A(f + \lambda g)) \in \Gamma(A)$. Therefore $\Gamma(A)$ is a vector space also.

5. Let (x_n, Ax_n) be a Cauchy sequence in $\Gamma(A)$. Then x_n is a Cauchy sequence in X (you show). So $x_n \to x \in X$. Because A is bounded, we know that A is continuous. Therefore $Ax_n \to Ax \in Y$. It follows that $(x_n, Ax_n) \to (x, Ax) \in \Gamma(A)$, and hence that $\Gamma(A)$ is a closed set.

6. (a) Since $\frac{1}{p} + \frac{1}{q} = 1$, it follows that $q = p(q-1)$. Then $\|a\|_p^p = \sum |a_j|^p = \sum \left| \frac{|b_j|^q}{b_j} \right|^p = \sum |b_j|^{(q-1)p} = \sum |b_j|^q = \|b\|_q^q < \infty$ because $b \in \ell^q(\mathbb{N})$. So $a \in \ell^p(\mathbb{N})$.
(b) Note first that $q - 1 = \frac{q}{p}$. So $\|b\|_q^{q-1} = \|b\|_q^{\frac{q}{p}} = \left(\|b\|_q^q\right)^{\frac{1}{p}} = \left(\|a\|_p^p\right)^{\frac{1}{p}} = \|a\|_p$ by the proof of part (a).
(c) Let $y \in \ell^q(\mathbb{N})$. By the Hölder Inequality, $|l_y(x)| = |\sum y_j x_j| \leq \sum |y_j x_j| = \|yx\|_1 \leq \|y\|_q \|x\|_p$ for all $x \in \ell^p(\mathbb{N})$. Therefore $\|l_y\| \leq \|y\|_q$.
(d) Let $y \in \ell^q(\mathbb{N})$. Choose $x \in \ell^p(\mathbb{N})$ such that $x_n y_n = |y_n|^q$. We can always do this by part (a). Then $|l_y(x)| = |\sum y_j x_j| = \sum |y_j|^q = \|y\|_q^q = \|y\|_q \|y\|_q^{q-1} = \|y\|_q \|x\|_p$ by part (b). It follows that $\|l_y\| = \sup_{\|x\|_p = 1} |l_y(x)| = \|y\|_q$.
(e) We showed in part (c) that l_y is bounded. It is easy to show that l_y is linear. Therefore $l_y \in \ell^p(\mathbb{N})^*$ for all $y \in \ell^q(\mathbb{N})$.