2016–17 **MAT462 Fonksiyonel Analiz II – Ödev 4** N. Course

SON TESLİM TARİHİ: Pazartesi 8 Mayıs 2017 saat 17:00'e kadar.

Egzersiz 9 (Compact Operators and Finite Rank Operators). Let X be a Hilbert space

(a) [10p] Let $K \in \mathcal{B}(X)$. Show that K^*K is symmetrical.

(b) [20p] Let $A \in \mathcal{B}(X)$. Show that $\operatorname{Ker}(A^*A) = \operatorname{Ker}(A)$.

(c) [20p] Let $T \in \mathcal{B}(X)$. Show that

$$T \text{ is a finite rank operator } \iff T^* \text{ is a finite rank operator.}$$

Egzersiz 10 (Fredholm Theory). Let X be a Hilbert space. Let $u, v \in X$ be any two vectors that satisfy $\langle u, v\rangle = 1$. Define an operator $K : X \to X$ by $K = \langle v, \cdot\rangle\, u$.

(a) [20p] Calculate $\operatorname{Ker}(1 - K)$.

(b) [10p] Calculate K^*

(c) [20p] Calculate $\operatorname{Ran}(1 - K)^\perp$.

Ödev 3'ün çözümleri

7. (a) Let $x_0 \in \overline{Y}$ and let $x_n \in Y$ be a sequence converging to x_0. Let $l \in Q$. Clearly $l(x_n) = 0$ for all n. By continuity it follows that $l(x_0) = l(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} l(x_n) = \lim_{n\to\infty} 0 = 0$ also.

   (b) Now suppose that $x_0 \in X \setminus \overline{Y}$. Then by Corollary 4.12, there exists $l \in Q$ such that $l(x_0) = \operatorname{dist}(x_0, Y) > 0$. The result follows by contraposition.

8. (a) Clearly $\|\delta^n\|_\infty = \sup_j \left|\delta^n_j\right| = |\delta^n_n| = 1 < \infty$. So $\delta^n \in \ell^\infty(\mathbb{N})$ for all n. Let $1 \leq p < \infty$. Then $\|\delta^n\|_p^p = \sum_{j=1}^\infty \left|\delta^n_j\right|^p = |\delta^n_n|^p = 1 < \infty$. So $\delta^n \in \ell^p(\mathbb{N})$ for all n.

   (b) Clearly $\|\delta^n - 0\|_p = 1 \not\to 0$ for all $1 \leq p \leq \infty$. Therefore $\delta^n \not\to 0$.

   (c) Let $l \in \ell^p(\mathbb{N})^*$. Then $\exists\ y \in \ell^q(\mathbb{N})$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that $l(x) = \sum_{j=1}^\infty y_j x_j$ for all $x \in \ell^p(\mathbb{N})$. Therefore $|l(\delta^n)| = \left|\sum_{j=1}^\infty y_j \delta^n_j\right| = |y_n| \to 0$ as $n \to \infty$, since $y \in \ell^q(\mathbb{N})$. Therefore $\delta^n \rightharpoonup 0$ as $n \to \infty$.

   (d) I will use proof by contradiction: Suppose that $\exists\ b$ such that $\delta^n \rightharpoonup b$ in $\ell^1(\mathbb{N})$. Thus $l(\delta^n) \to l(b)$ for all $l \in \ell^1(\mathbb{N})^*$

   Consider first the functional $\tilde{l} \in \ell^1(\mathbb{N})^*$ defined by $\tilde{l}(x) = x_1 + x_2 + x_3 + x_4 + \ldots$. Then $\tilde{l}(\delta^n) = 1$ for all $n \in \mathbb{N}$, so

   $$b_1 + b_2 + b_3 + \ldots = \tilde{l}(b) = \lim_{n\to\infty} l(\delta^n) = \lim_{n\to\infty} 1 = 1 \tag{1}$$

   Now define a family of linear functionals $l_j(x) = x_j$. Clearly l_j is bounded and linear for all j. However $b_j = l_j(b) = \lim_{n\to\infty} l_j(\delta^n) = \lim_{n\to\infty} \delta^n_j = 0$ for all j, which contradicts (1).