OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

24.05.2012 **MAT 462 – Fonksiyonel Analiz II – Yarıyıl Sonu Sınavı** N. Course

ADI SOYADI
ÖĞRENCİ NO
İMZA

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have 120 minutes to answer 4 questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You should write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam 120 dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan 4 tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	4	5	TOTAL

Notation:

$$
\begin{aligned}
C([a,b]) &= \{f : [a,b] \to \mathbb{C} : f \text{ is continuous }\} \\
C^1([a,b]) &= \{f : [a,b] \to \mathbb{C} : f \text{ and } f' \text{ are continuous }\} \\
C^\infty([a,b]) &= \{f : [a,b] \to \mathbb{C} : \tfrac{d^n f}{dx^n} \text{ exists and is continuous } \forall n\}
\end{aligned}
$$

$$
\begin{aligned}
\|f\|_\infty &= \max_{x\in[0,1]} |f(x)| \\
\|f\|_{\infty,1} &= \|f\|_\infty + \|f'\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^2_{cont}([a,b]) &= \big(C([a,b]), \langle\cdot,\cdot\rangle_{L^2}\big) \\
\langle f,g\rangle_{L^2} &= \int_a^b \overline{f(x)} g(x)\ dx
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}(X,Y) &= \{A : X \to Y : A \text{ is linear and bounded}\} \\
\mathcal{B}(X) &= \mathcal{B}(X,X) \\
\mathcal{K}(X,Y) &= \{A : X \to Y : A \text{ is linear and compact}\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{x+iy} &= x - iy \\
A^* &= \text{adjoint of } A \\
\mathrm{Ker}(A) &= \text{kernal of } A = \{f \in X : Af = 0\} \\
\mathrm{Ran}(A) &= \text{range of } A = \{Af : f \in X\} \\
M^\perp &= \text{orthogonal complement of } M
\end{aligned}
$$

$$
\begin{aligned}
X^* &= \text{dual space of } X \\
X^{**} &= \text{double dual space of } X
\end{aligned}
$$

$$
\begin{aligned}
\ell^p(\mathbb{N})^* &\cong \ell^q(\mathbb{N}) \qquad 1 \le p < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1 \\
\ell^\infty(\mathbb{N})^* &\ncong \ell^1(\mathbb{N})
\end{aligned}
$$

Question 1 (Weak Convergence).

(a) [5 pts] Let X be a Banach space. Give the definition of *weak convergence* in X [i.e. $x_n \rightharpoonup x$ for $x_n \in X$.].

Consider the Banach space $\ell^p(\mathbb{N})$ where

$$\ell^p(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \|a\|_p := \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$, and

$$\ell^\infty(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \|a\|_\infty := \sup_j |a_j| < \infty \right\}.$$

Define

$$\delta_j^n = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j. \end{cases}$$

[For example, δ^5 is the sequence $(0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)$]

(b) [6 pts] Show that $\delta^n \in \ell^p(\mathbb{N})$ for all $n \in \mathbb{N}$ and for all $1 \leq p \leq \infty$.

(c) [7 pts] Let $1 < p < \infty$. Show that $\delta^n \rightharpoonup 0$.

(d) [7 pts] Show that δ^n is not weakly convergent in $\ell^1(\mathbb{N})$.

ÖĞRENCİ NO.

Question 2 (Dual Spaces). Let X be a normed vector space.

(a) [4 pts] Give the definition of the *dual space* of X.

(b) [4 pts] Let $x_0 \in X$ and $Y \subseteq X$. Give the definition of

$$\operatorname{dist}(x_0, Y).$$

Corollary 4.12. *Let X be a normed vector space and let $Y \subseteq X$ be a subspace. Let $x_0 \in X \setminus \overline{Y}$. Then $\exists l \in X^*$ such that*

(i) $l(y) = 0 \quad \forall y \in Y$;

(ii) $l(x_0) = \operatorname{dist}(x_0, Y)$; *and*

(iii) $\|l\| = 1$.

(c) [17 pts] Let X be a normed vector space and let $Y \subseteq X$ be a subspace. Define

$$S := \{f \in X^* : f(y) = 0 \ \forall y \in Y\} \subseteq X^*.$$

Use Corollary 4.12 to prove that

$$x_0 \in \overline{Y} \qquad \Longleftrightarrow \qquad l(x_0) = 0 \quad \forall l \in S.$$

Question 3 (Weak and Strong Convergence of Operators)**.** Consider the Hilbert space $\ell^2(\mathbb{N}) = \left\{a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \|a\|_2 < \infty\right\}$ with the inner product $\langle x, y\rangle_2 = \sum_{j=1}^{\infty} \overline{x_j} y_j$.

Define two sequences of (bounded linear) operators $S_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ and $S_n^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S_n(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)$$

and

$$S_n^*(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (\underbrace{0, 0, \ldots, 0}_{n \text{ terms}}, x_1, x_2, x_3, x_4, \ldots).$$

In other words, S_n shifts every term, n places to the left; and S_n^* shifts every term, n places to the right.

(a) [5 pts] Show that $\|S_n\| = 1, \ \forall n \in \mathbb{N}$.

(b) [5 pts] Show that $\|S_n^*\| = 1, \ \forall n \in \mathbb{N}$.

(c) [5 pts] Show that S_n^* is the adjoint of S_n.

[HINT: In other words, show that $\langle x, S_n^* y\rangle_2 = \langle S_n x, y\rangle_2$ for all $x, y \in \ell^2(\mathbb{N})$.]

(d) [5 pts] Show that $S_n \not\to 0$ as $n \to \infty$.

(e) [5 pts] Show that $\text{s-lim}_{n\to\infty} S_n = 0$.

Question 4 (Closed Operators)**.** Let X and Y be Banach spaces.

(a) [4 pts] Give the definition of the *graph* of an operator $A : \mathfrak{D}(A) \subseteq X \to Y$.

(b) [4 pts] Give the definition of a *closed operator*.

(c) [7 pts] Now suppose that $A : X \to Y$ is a bounded operator. Show that A is a closed operator.
[HINT: Start by letting (x_n, Ax_n) be any Cauchy sequence in $\Gamma(A)$.]

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach space. Let $A : \mathfrak{D}(A) \subseteq X \to Y$ be an operator. We can define a new norm, called the *graph norm associated with* A, by

$$\|x\|_A = \|x\|_X + \|Ax\|_Y$$

for all $x \in \mathfrak{D}(A)$.

(d) [10 pts] Show that $A : (\mathfrak{D}(A), \|\cdot\|_A) \to (Y, \|\cdot\|_Y)$ is bounded.

Question 5 (Compact Operators)**.** Let X be a Hilbert space.

(a) [5 pts] Give the definition of a *compact operator*.

Let $K \in \mathcal{K}(X)$ be compact. Let s_j be the singular values of K and let $\{u_j\}$ be the corresponding orthonormal eigenvectors of K^*K. Then

$$K = \sum_j s_j \langle u_j, \cdot \rangle v_j$$

where

$$v_j = \frac{1}{s_j} K u_j$$

by Theorem 5.1.

(b) [10 pts] Show that $\|K\| \leq \max_j \{s_j\}$.

(c) [10 pts] Show that $\|K\| \geq \max_j \{s_j\}$.