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MATEMATİK BÖLÜMÜ

24.05.2012**MAT 462 – Fonksiyonel Analiz II – Yarıyıl Sonu Sınavı Çözümleri** N. Course

Question 1 (Weak Convergence)**.**

(a) [5 pts] Let X be a Banach space. Give the definition of *weak convergence* in X [i.e. $x_n \rightharpoonup x$ for $x_n \in X$.].

> We say that x_n converges weakly to x, and write $x_n \rightharpoonup x$, iff $l(x_n) \to l(x)$ for all $l \in X^*$.

Consider the Banach space $\ell^p(\mathbb{N})$ where

$$\ell^p(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_p := \left(\sum_{j=1}^\infty |a_j|^p \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$, and

$$\ell^\infty(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_\infty := \sup_j |a_j| < \infty \right\}.$$

Define

$$\delta_j^n = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j. \end{cases}$$

[For example, δ^5 is the sequence $(0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)$]

(b) [6 pts] Show that $\delta^n \in \ell^p(\mathbb{N})$ for all $n \in \mathbb{N}$ and for all $1 \leq p \leq \infty$.

> Clearly
> $$\|\delta^n\|_\infty = \sup_j \left| d_j^n \right| = |\delta_n^n| = 1 < \infty.$$
> So $\delta^n \in \ell^\infty(\mathbb{N})$ for all n.
> Let $1 \leq p < \infty$. Then
> $$\|\delta^n\|_p^p = \sum_{j=1}^\infty \left|\delta_j^n\right|^p = \|\delta_n^n\| = 1 < \infty.$$
> So $\delta^n \in \ell^p(\mathbb{N})$ for all n.

(c) [7 pts] Let $1 < p < \infty$. Show that $\delta^n \rightharpoonup 0$.

> Let $l \in \ell^p(\mathbb{N})^*$. Then $\exists\ y \in \ell^q(\mathbb{N})$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that
> $$l(x) = \sum_{j=1}^\infty y_j x_j$$
> for all $x \in \ell^p(\mathbb{N})$.
> So
> $$|l(\delta^n)| = \left| \sum_{j=1}^\infty y_j \delta_j^n \right| = |y_n| \to 0$$
> as $n \to \infty$ since $y \in \ell^q(\mathbb{N})$.
> Therefore $\delta \rightharpoonup 0$ as $n \to \infty$.

(d) [7 pts] Show that δ^n is not weakly convergent in $\ell^1(\mathbb{N})$.

Consider first the functional $l \in \ell^1(\mathbb{N})^*$ defined by $l(x) = x_1$. Then $l(\delta^n) = 0$ for all $n \geq 2$. So clearly $l(\delta^n) \to 0 = l(0)$. Therefore, *if* δ^n is weakly convergent in $\ell^1(\mathbb{N})$, then $\delta^n \rightharpoonup 0$.
Next consider $\tilde{l} \in \ell^1(\mathbb{N})^*$ defined by $\tilde{l}(x) = x_1 + x_2 + x_3 + x_4 + \ldots$. Then $\tilde{l}(\delta^n) = 1$ for all $n \in \mathbb{N}$, so $\tilde{l}(\delta^n) \not\to 0$ as $n \to \infty$.
Therefore δ^n is not weakly convergent in $\ell^1(\mathbb{N})$.

Question 2 (Dual Spaces)**.** Let X be a normed vector space.

(a) [4 pts] Give the definition of the *dual space* of X.

The dual space of X is $X^* = \mathcal{B}(X, \mathbb{C})$ [or $\mathcal{B}(X, \mathbb{R})$ if X is a real vector space].

(b) [4 pts] Let $x_0 \in X$ and $Y \subseteq X$. Give the definition of

$$\operatorname{dist}(x_0, Y).$$

$$\operatorname{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\|_X$$

Corollary 12. *Let X be a normed vector space and let $Y \subseteq X$ be a subspace. Let $x_0 \in X \setminus \overline{Y}$. Then $\exists l \in X^*$ such that*

(i) $l(y) = 0 \quad \forall y \in Y$*;*

(ii) $l(x_0) = \operatorname{dist}(x_0, Y)$*; and*

(iii) $\|l\| = 1$.

(c) [17 pts] Let X be a normed vector space and let $Y \subseteq X$ be a subspace. Define

$$S := \{f \in X^* : f(y) = 0 \ \forall y \in Y\} \subseteq X^*.$$

Use Corollary 4.12 to prove that

$$x_0 \in \overline{Y} \qquad \iff \qquad l(x_0) = 0 \ \ \forall l \in S.$$

"$\Longrightarrow$"
Let $x_0 \in \overline{Y}$. Then $\exists$ a sequence $(x_n) \subseteq Y$ such that $x_n \to x_0$. Let $l \in S$. Then $l(x_n) = 0 \ \forall n$. It follows by continuity that $l(x_0) = 0$ also. $\boxed{8}$

"$\Longleftarrow$"
Now suppose that $x_0 \notin \overline{Y}$. By Corollary 4.12, $\exists l \in X^*$ such that $l(y) = 0 \ \forall y \in Y$ (i.e. $l \in S$) and $l(x_0) = \operatorname{dist}(x_0, Y) > 0$. Therefore $\exists l \in S$ such that $l(x_0) \neq 0$, and we are finished. $\boxed{9}$

Question 3 (Weak and Strong Convergence of Operators)**.** Consider the Hilbert space $\ell^2(\mathbb{N}) = \left\{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_2 < \infty\right\}$ with the inner product $\langle x, y \rangle_2 = \sum_{j=1}^\infty \overline{x_j} y_j$.

Define two sequences of (bounded linear) operators $S_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ and $S_n^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S_n(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)$$

and

$$S_n^*(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (\underbrace{0, 0, \ldots, 0}_{n \text{ terms}}, x_1, x_2, x_3, x_4, \ldots).$$

In other words, S_n shifts every term, n places to the left; and S_n^* shifts every term, n places to the right.

(a) [5 pts] Show that $\|S_n\| = 1, \ \forall n \in \mathbb{N}$.

First $\|S_n x\|_2 = (\sum_{j=1}^{\infty} |(S_n x)_j|^2)^{\frac{1}{2}} = (\sum_{j=n+1}^{\infty} |x_j|^2)^{\frac{1}{2}} \leq (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}} = \|x\|_2$ for all x. So $\|S_n\| \leq 1$. Moreover, $\|\delta^m\|_2 = 1$ for all $m \in \mathbb{N}$, and $\|S_n \delta^{n+1}\|_2 = \|\delta^1\|_2 = 1 = \|\delta^{n+1}\|_2$. Therefore $\|S_n\| \geq 1$.

(b) [5 pts] Show that $\|S_n^*\| = 1, \ \forall n \in \mathbb{N}$.

$\|S_n^* x\|_2 = (\sum_{j=1}^{\infty} |(S_n^* x)_j|^2)^{\frac{1}{2}} = (\sum_{j=n+1}^{\infty} |(S_n^* x)_j|^2)^{\frac{1}{2}} = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}} = \|x\|_2$ for all x. Therefore $\|S_n^*\| = 1$.

(c) [5 pts] Show that S_n^* is the adjoint of S_n.
[HINT: In other words, show that $\langle x, S_n^* y\rangle_2 = \langle S_n x, y\rangle_2$ for all $x, y \in \ell^2(\mathbb{N})$.]

$\langle x, S_n^* y\rangle = \sum_{j=1}^{\infty} \overline{x_j}(S_n^* y)_j = \sum_{j=n+1}^{\infty} \overline{x_j} y_{j-n} = \sum_{j=1}^{\infty} \overline{x_{n+j}} y_j = \sum_{j=1}^{\infty} \overline{(S_n x)_j} y_j = \langle S_n x, y\rangle$

(d) [5 pts] Show that $S_n \not\to 0$ as $n \to \infty$.

Since $\|S_n\| = 1$ for all n, it follows that $\|S_n - 0\| \not\to 0$. So $S_n \not\to 0$.

(e) [5 pts] Show that $\text{s-lim}_{n\to\infty} S_n = 0$.

Let $x \in \ell^2(\mathbb{N})$. Then $\|x\|_2 < \infty$. So $\sum_{j=1}^{n} |x_j|^2 \to \sum_{j=1}^{\infty} |x_j|^2$ as $n \to \infty$. Therefore

$$\|S_n x\|_2 = (\sum_{j=1}^{\infty} |(S_n x)_j|^2)^{\frac{1}{2}} = (\sum_{j=n+1}^{\infty} |x_j|^2)^{\frac{1}{2}} \to 0$$

as $n \to \infty$. Hence $S_n x \to 0$ for all x, and thus $\text{s-lim}_{n\to\infty} S_n = 0$.

Question 4 (Closed Operators)**.** Let X and Y be Banach spaces.

(a) [4 pts] Give the definition of the *graph* of an operator $A : \mathfrak{D}(A) \subseteq X \to Y$.

$$\Gamma(A) = \{(x, Ax) : x \in \mathfrak{D}(A)\}$$

(b) [4 pts] Give the definition of a *closed operator*.

An operator is called closed iff its graph is a closed set.

(c) [7 pts] Now suppose that $A : X \to Y$ is a bounded operator. Show that A is a closed operator.
[HINT: Start by letting (x_n, Ax_n) be any Cauchy sequence in $\Gamma(A)$.]

Let (x_n, Ax_n) be a Cauchy sequence in $\Gamma(A) \subseteq X \oplus Y$. Then x_n is a Cauchy sequence in the Banach space X. So $x_n \to x \in X$.
Now, A is bounded $\boxed{1}$, so A is continuous $\boxed{1}$. Therefore $x_n \to x \implies Ax_n \to Ax$. So $(x_n, Ax_n) \to (x, Ax) \in \Gamma(A)$. So $\Gamma(A)$ is closed.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach space. Let $A : \mathfrak{D}(A) \subseteq X \to Y$ be an operator. We can define a new norm, called the *graph norm associated with* A, by

$$\|x\|_A = \|x\|_X + \|Ax\|_Y$$

for all $x \in \mathfrak{D}(A)$.

(d) [10 pts] Show that $A : (\mathfrak{D}(A), \|\cdot\|_A) \to (Y, \|\cdot\|_Y)$ is bounded.

Let $x \in \mathfrak{D}(A)$. Then

$$\|Ax\|_Y \leq \|x\|_X + \|Ax\|_Y = \|x\|_A \,.$$

Therefore $\|A\| = \sup_{\|x\|_A = 1} \|Ax\|_Y \leq 1 < \infty$. Therefore A is bounded.

Question 5 (Compact Operators)**.** Let X be a Hilbert space.

(a) [5 pts] Give the definition of a *compact operator*.

An operator $A : X \to Y$ is called compact iff,

$$(f_n) \subseteq X \text{ bounded} \implies (Af_n) \subseteq Y \text{ has a convergent subsequence.}$$

Let $K \in \mathcal{K}(X)$ be compact. Let s_j be the singular values of K and let $\{u_j\}$ be the corresponding orthonormal eigenvectors of K^*K. Then

$$K = \sum_j s_j \langle u_j, \cdot \rangle v_j$$

where

$$v_j = \frac{1}{s_j} K u_j$$

by Theorem 5.1.

(b) [10 pts] Show that $\|K\| \leq \max_j\{s_j\}$.

$$\begin{aligned}
\|Kf\|^2 &= \left\| \sum_j s_j \langle u_j, f \rangle v_j \right\|^2 \\
&= \sum_j \|s_j \langle u_j, f \rangle v_j\|^2 && \text{(since the } v_j \text{ are orthogonal)} \\
&= \sum_j |s_j|^2 \, |\langle u_j, f \rangle|^2 \\
&\leq \max_j\{s_j\} \sum_j |\langle u_j, f \rangle|^2 && \text{(since the } s_j \text{ are real and positive)} \\
&= \max_j\{s_j\} \, \|f\|^2 .
\end{aligned}$$

Therefore $\|K\| \leq \max_j\{s_j\}$.

(c) [10 pts] Show that $\|K\| \geq \max_j\{s_j\}$.

Finally, choose j_0 such that $s_{j_0} = \max_j\{s_j\}$. Then

$$\begin{aligned}
\|Ku_{j_0}\| &= \left\| \sum_j s_j \langle u_j, u_{j_0} \rangle v_j \right\|^2 \\
&= \|s_{j_0} v_{j_0}\| = s_{j_0} .
\end{aligned}$$

Therefore

$$\|K\| = \sup_{\|f\|=1} \|Kf\| \geq \|Ku_{j_0}\| = s_{j_0} = \max_j\{s_j\}.$$