OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2013.05.23 **MAT 462 – Fonksiyonel Analiz II – Final Sınavı** N. Course

ADI: Ö R N E K T İ R

SOYADI: S A M P L E

ÖĞRENCİ NO: 0 1 0 6 0

İMZA:

Süre: **120** dk.

Bu sorulardan **4** tanesini seçerek cevaplayınız.

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have **120** minutes to answer **4** questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You must show your working for all questions.
5. Write your student number on every page.
6. This exam contains 12 pages. Check to see if any pages are missing.
7. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the first 20 minutes, or in the final 10 minutes, of the exam.
8. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
9. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
10. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam **120** dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan **4** tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Sonuca ulaşmak için yaptığınız işlemleri ayrıntılarıyla gösteriniz.
5. Öğrenci numaranızı her sayfaya yazınız.
6. Sınav 12 sayfadan oluşmaktadır. Lütfen eksik sayfa olup olmadığını kontrol edin.
7. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın ilk 20 dakikası ve son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
8. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
9. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
10. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	4	5	TOPLAM

Notation:

$$
\begin{aligned}
C([a,b]) &= \{f : [a,b] \to \mathbb{C} : f \text{ is continuous }\} \\
C^1([a,b]) &= \{f : [a,b] \to \mathbb{C} : f \text{ and } f' \text{ are continuous }\} \\
C^\infty([a,b]) &= \{f : [a,b] \to \mathbb{C} : \tfrac{d^n f}{dx^n} \text{ exists and is continuous } \forall n\}
\end{aligned}
$$

$$
\begin{aligned}
\|f\|_\infty &= \max_{x\in[0,1]} |f(x)| \\
\|f\|_{\infty,1} &= \|f\|_\infty + \|f'\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^2_{cont}([a,b]) &= \big(C([a,b]), \langle\cdot,\cdot\rangle_{L^2}\big) \\
\langle f,g\rangle_{L^2} &= \int_a^b \overline{f(x)} g(x)\; dx
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}(X,Y) &= \{A : X \to Y : A \text{ is linear and bounded}\} \\
\mathcal{B}(X) &= \mathcal{B}(X,X) \\
\mathcal{K}(X,Y) &= \{A : X \to Y : A \text{ is linear and compact}\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{x+iy} &= x - iy \\
A^* &= \text{adjoint of } A \\
\text{Ker}(A) &= \text{kernal of } A = \{f \in X : Af = 0\} \\
\text{Ran}(A) &= \text{range of } A = \{Af : f \in X\} \\
M^\perp &= \text{orthogonal complement of } M
\end{aligned}
$$

$$
\begin{aligned}
X^* &= \text{dual space of } X \\
X^{**} &= \text{double dual space of } X
\end{aligned}
$$

$$
\begin{aligned}
\ell^p(\mathbb{N})^* &\cong \ell^q(\mathbb{N}) \qquad 1 \le p < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1 \\
\ell^\infty(\mathbb{N})^* &\ncong \ell^1(\mathbb{N})
\end{aligned}
$$

$$\sum_{j=1}^{n} |\langle f, u_j\rangle|^2 \le \|f\|^2 \qquad \text{Bessel's Inequality } (\{u_j\} \text{ orthonormal})$$

$$\|xy\|_1 \le \|x\|_p \|y\|_q \qquad \text{Hölder's Inequality } (\tfrac{1}{p} + \tfrac{1}{q} = 1)$$

$$|\langle f,g\rangle| \le \|f\| \, \|g\| \qquad \text{Cauchy-Schwarz Inequality}$$

Soru 1 (Fredholm Theory)**.** Let X be a Hilbert space. Let $K : X \to X$ be a compact operator (i.e. $K \in \mathcal{K}(X)$). Suppose that $\mathrm{Ker}(1-K) = \{0\}$.

(a) [10p] Suppose that $\mathrm{Ran}(1-K) \neq X$. Define $X_1 := \mathrm{Ran}(1-K) = (1-K)X \subsetneqq X$ and $X_2 := (1-K)X_1 \subseteq X_1$. Show that $X_1 \neq X_2$.

[HINT: Use proof by contradiction. Start with $X_1 = X_2$, $x \in X_1^\perp$, $x \neq 0$ and $y := (1-K)x$. Show that $\exists\, z \in X_1$ such that $(1-K)z = y$. Then prove that this contradicts $\mathrm{Ker}(1-K) = \{0\}$.]

Still assuming that $\mathrm{Ker}(1-K) = \{0\}$ and $\mathrm{Ran}(1-K) \neq X$, by repeating this idea, we can define

$$\begin{aligned}
X_1 &:= (1-K)X \subsetneqq X \\
X_2 &:= (1-K)X_1 = (1-K)^2 X \subsetneqq X_1 \\
X_3 &:= (1-K)X_2 = (1-K)^3 X \subsetneqq X_2 \\
X_4 &:= (1-K)X_3 = (1-K)^4 X \subsetneqq X_3 \\
X_5 &:= (1-K)X_4 = (1-K)^5 X \subsetneqq X_4 \\
&\vdots \\
X_j &:= (1-K)X_{j-1} = (1-K)^j X \subsetneqq X_{j-1} \\
&\vdots
\end{aligned}$$

which gives us a sequence of subspaces $X \supsetneqq X_1 \supsetneqq X_2 \supsetneqq X_3 \supsetneqq X_4 \subsetneqq X_5 \supsetneqq \dots$

For each j, choose $f_j \in X_j \cap X_{j+1}^{\perp}$ such that $\|f_j\| = 1$.

(b) [5p] Suppose $k > j$. Show that

$$f_k + (1 - K)(f_j - f_k) \in X_{j+1}.$$

(c) [5p] Show that

$$k > j \qquad \Longrightarrow \qquad \|Kf_j - Kf_k\|^2 \geq 1.$$

[HINT: Remember: If $\langle a, b \rangle = 0$, then $\|a + b\|^2 = \|a\|^2 + \|b\|^2$ by Pythogoras. $Kf_j = f_j - (1 - K)f_j$. Use part (b) and the fact that $\|f_j\|^2 = 1$.]

(d) [5p] Now prove that

$$\mathrm{Ker}(1 - K) = \{0\} \qquad \Longrightarrow \qquad \mathrm{Ran}(1 - K) = X.$$

[HINT: Use proof by contradiction and parts (a)-(c). Remember that K is compact – what do we know about the sequence (f_j)?]

Soru 2 (Weak and Strong Convergence of Operators)**.** Let X be a Banach space. Let $A_n, B_n \in \mathcal{B}(X)$ be 2 sequences of bounded operators

(a) [5p] Give the definition of "B_n *converges strongly to* B" [i.e. $\text{s-lim}_{n\to\infty} B_n = B$].

(b) [5p] Give the definition of "A_n *converges weakly to* A" [i.e. $\text{w-lim}_{n\to\infty} A_n = A$].

(c) [14p] Show that

$$\operatorname*{w\text{-}lim}_{n\to\infty} A_n = A \quad\text{and}\quad \operatorname*{s\text{-}lim}_{n\to\infty} B_n = B \qquad \Longrightarrow \qquad \operatorname*{w\text{-}lim}_{n\to\infty} A_n B_n = AB.$$

(d) [1p] Is the following statement true or false?

$$\text{“}\operatorname*{w\text{-}lim}_{n\to\infty} A_n = A \text{ and } \operatorname*{s\text{-}lim}_{n\to\infty} B_n = B \qquad \Longrightarrow \qquad \operatorname*{w\text{-}lim}_{n\to\infty} B_n A_n = BA.\text{”}$$

☐ true ☐ false

Soru 3 (Dual Space)**.** Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider the Banach spaces $\ell^p(\mathbb{N})$, $\ell^q(\mathbb{N})$ and $\ell^p(\mathbb{N})^*$, where

$$\ell^p(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \|a\|_p := \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $b = (b_j)_{j=1}^{\infty} \in \ell^q(\mathbb{N})$. Define

$$a_j = \begin{cases} \frac{|b_j|^q}{b_j} & \text{if } b_j \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) [5p] Show that $a = (a_j)_{j=1}^{\infty} \in \ell^p(\mathbb{N})$.
[HINT: $\frac{1}{p} + \frac{1}{q} = 1 \iff \frac{q+p}{pq} = 1 \iff \ldots$. Show first that $\|a\|_p^p = \|b\|_q^q$.]

(b) [5p] Show that $\|b\|_q^{q-1} = \|a\|_p$.

For each $y \in \ell^q(\mathbb{N})$, define $l_y : \ell^p(\mathbb{N}) \to \mathbb{C}$ by

$$l_y(x) = \sum_{j=1}^{\infty} y_j x_j.$$

(c) [5p] Use the Hölder Inequality to show that $\|l_y\| \leq \|y\|_q$ for all $y \in \ell^q(\mathbb{N})$.

(d) [8p] Show that $\|l_y\| = \|y\|_q$ for all $y \in \ell^q(\mathbb{N})$.
[HINT: Choose $x \in \ell^p(\mathbb{N})$ such that $x_j y_j = |y_j|^q$. Why can we always do this? Use part (b).]

(e) [2p] Show that $l_y \in \ell^p(\mathbb{N})^*$ for all $y \in \ell^q(\mathbb{N})$.

Soru 4 (Closed Operators)**.** Let X and Y be a Banach spaces.

(a) [4p] Give the definition of the *graph* of an operator $A : \mathfrak{D}(A) \subseteq X \to Y$.

(b) [4p] Give the definition of a *closed operator*.

(c) [8p] Now let $A : X \to Y$ be an operator. Suppose that A satisfies the following property:

- Let (x_n) be any sequence in X. If $x_n \to x$ and $Ax_n \to y$, then $Ax = y$.

Show that A is a closed operator.
[HINT: Start by letting (x_n, Ax_n) be any Cauchy sequence in $\Gamma(A)$.]

(d) [8p] Now let X be a Hilbert space. Let $A : X \to X$ be a symmetrical operator [i.e. $\langle x, Ay \rangle = \langle Ax, y \rangle \; \forall x, y \in X$]. Let (x_n) be a sequence such that $x_n \to x \in X$ and $Ax_n \to y \in X$.

Show that $Ax = y$.

(e) [1p] Is the following statement true or false?

"Every symmetrical operator, defined on a Hilbert space, is a closed operator."

☐ true ☐ false

Soru 5 (Weak Convergence)**.**

(a) [5p] Let X be a Banach space. Give the definition of *weak convergence* in X [i.e. $x_n \rightharpoonup x$ for $x_n \in X$.].

Now let X be a Hilbert space. Let $\{u_j\}_{j=1}^{\infty} \subseteq X$ be a countable, infinite, orthonormal set.

(b) [10p] Show that

$$\langle g, u_n \rangle \to 0$$

as $n \to \infty$, for all $g \in X$.

(c) [5p] Show that $u_n \rightharpoonup 0$ as $n \to \infty$.

(d) [5p] Show that $u_n \not\to 0$ as $n \to \infty$.