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Question 1 (Fredholm Theory)**.** Let X be a Hilbert space. Let $K : X \to X$ be a compact operator (i.e. $K \in \mathcal{K}(X)$). Suppose that $\mathrm{Ker}(1-K) = \{0\}$.

(a) [10p] Suppose that $\mathrm{Ran}(1-K) \neq X$. Define $X_1 := \mathrm{Ran}(1-K) = (1-K)X \subsetneqq X$ and $X_2 := (1-K)X_1 \subseteq X_1$. Show that $X_1 \neq X_2$.

[HINT: Use proof by contradiction. Start with $X_1 = X_2$, $x \in X_1^\perp$, $x \neq 0$ and $y := (1-K)x$. Show that $\exists\ z \in X_1$ such that $(1-K)z = y$. Then prove that this contradicts $\mathrm{Ker}(1-K) = \{0\}$.]

> Proof by contradiction: Suppose that $X_1 = X_2$.
>
> Let $x \in X_1^\perp$ such that $x \neq 0$ (this is possible since we know that $X_1 \neq X$ $\boxed{1}$). Define $y = (1-K)x$.
>
> Then $y \in \mathrm{Ran}(1-K) = X_1 = X_2 = (1-K)X_1$ $\boxed{2}$. So $\exists z \in X_1$ such that $y = (1-K)z$ $\boxed{2}$. But then $(1-K)x = y = (1-K)z$ which implies that $(1-K)(x-z) = 0$. So $x - z \in \mathrm{Ker}(1-K)$ $\boxed{2}$. Given that $x \in X_1^\perp$, $z \in X_1$ and $x \neq 0$, we know that $x - z \neq 0$ $\boxed{2}$. However $\mathrm{Ker}(1-K) = \{0\}$. Contradiction. $\boxed{1}$

Still assuming that $\mathrm{Ker}(1-K) = \{0\}$ and $\mathrm{Ran}(1-K) \neq X$, by repeating this idea, we can define

$$
\begin{aligned}
X_1 &:= (1-K)X \subsetneqq X \\
X_2 &:= (1-K)X_1 = (1-K)^2 X \subsetneqq X_1 \\
X_3 &:= (1-K)X_2 = (1-K)^3 X \subsetneqq X_2 \\
X_4 &:= (1-K)X_3 = (1-K)^4 X \subsetneqq X_3 \\
X_5 &:= (1-K)X_4 = (1-K)^5 X \subsetneqq X_4 \\
&\ \vdots \\
X_j &:= (1-K)X_{j-1} = (1-K)^j X \subsetneqq X_{j-1} \\
&\ \vdots
\end{aligned}
$$

which gives us a sequence of subspaces $X \supsetneqq X_1 \supsetneqq X_2 \supsetneqq X_3 \supsetneqq X_4 \subsetneqq X_5 \supsetneqq \ldots$

For each j, choose $f_j \in X_j \cap X_{j+1}^\perp$ such that $\|f_j\| = 1$.

(b) [5p] Suppose $k > j$. Show that

$$f_k + (1-K)(f_j - f_k) \in X_{j+1}.$$

> Clearly $(1-K)f_j \in (1-K)X_j = X_{j+1}$ $\boxed{1}$. Moreover $f_k \in X_k \subseteq X_{j+1}$ $\boxed{1}$ and $(1-K)f_k \in (1-K)X_k \subsetneqq X_k \subseteq X_{j+1}$ $\boxed{2}$. Therefore $f_k + (1-K)(f_j - f_k) \in X_{j+1}$. $\boxed{1}$

(c) [5p] Show that

$$k > j \qquad \Longrightarrow \qquad \|Kf_j - Kf_k\|^2 \geq 1.$$

[HINT: Remember: If $\langle a, b\rangle = 0$, then $\|a+b\|^2 = \|a\|^2 + \|b\|^2$ by Pythogoras. $Kf_j = f_j - (1-K)f_j$. Use part (b) and the fact that $\|f_j\|^2 = 1$.]

Suppose that $k > j$. Then

$$\|Kf_j - Kf_k\|^2 = \|f_j - f_k - (1-K)(f_j - f_k)\|^2 = \|f_j\|^2 + \|f_k + (1-K)(f_j - f_k)\|^2 \geq 1$$

by Pythogoras, since $f_j \in X_{j+1}^{\perp}$ and $f_k + (1-K)(f_j - f_k) \in X_{j+1}$.

(d) [5p] Now prove that

$$\operatorname{Ker}(1-K) = \{0\} \qquad \Longrightarrow \qquad \operatorname{Ran}(1-K) = X.$$

[HINT: Use proof by contradiction and parts (a)-(c). Remember that K is compact – what do we know about the sequence (f_j)?]

Proof by contradiction: Suppose that $\operatorname{Ker}(1-K) = \{0\}$ and $\operatorname{Ran}(1-K) \neq X$.

Let (f_j) be the bounded sequence defined above. Because K is compact, we know that (Kf_j) has a convergence subsequence. But this contradicts part (c).

Question 2 (Weak and Strong Convergence of Operators)**.** Let X be a Banach space. Let $A_n, B_n \in \mathcal{B}(X)$ be 2 sequences of bounded operators

(a) [5p] Give the definition of "B_n *converges strongly to* B" [i.e. $\text{s-lim}_{n\to\infty} B_n = B$].

A sequence of operators (B_n) is said to converge strongly to B iff, $B_n x \to Bx$ for all $x \in \mathfrak{D}(B) \subseteq \mathfrak{D}(B_n)$.

(b) [5p] Give the definition of "A_n *converges weakly to* A" [i.e. $\text{w-lim}_{n\to\infty} A_n = A$].

A sequence of operators (A_n) is said to converge weakly to A iff, $A_n x \rightharpoonup Ax$ for all $x \in \mathfrak{D}(A) \subseteq \mathfrak{D}(A_n)$.

(c) [14p] Show that

$$\operatorname*{w-lim}_{n\to\infty} A_n = A \ \text{ and } \ \operatorname*{s-lim}_{n\to\infty} B_n = B \qquad \Longrightarrow \qquad \operatorname*{w-lim}_{n\to\infty} A_n B_n = AB.$$

Let $x \in \mathfrak{D}(A) \subseteq \mathfrak{D}(A_n)$ and define $y := Bx$. Let $l \in X^*$.

Since $\text{w-lim}_{n\to\infty} A_n = A$, we know that $A_n y \rightharpoonup Ay$, which tells us that $l(A_n y) \to l(Ay)$. Moreover, since $\text{s-lim}_{n\to\infty} B_n = B$, we know that $B_n x \to Bx$. Then

$$\begin{aligned}
\big\| l\big((A_nB_n - AB)x\big)\big\| &= \|lA_nB_nx - lABx\| \\
&= \|lA_nB_nx - lA_nBx + lA_nBx - lABx\| \\
&\leq \|lA_nB_nx - lA_nBx\| + \|lA_nBx - lABx\| \\
&\leq \|l\|\,\|A\|\,\|B_nx - Bx\| + \|l(A_ny - Ay)\| \\
&\to 0.
\end{aligned}$$

Therefore $A_nB_nx \rightharpoonup ABx$ and hence $\text{w-lim}_{n\to\infty} A_nB_n = AB$.

(d) [1p] Is the following statement true or false?

$$\text{“}\operatorname*{w-lim}_{n\to\infty} A_n = A \text{ and } \operatorname*{s-lim}_{n\to\infty} B_n = B \qquad \Longrightarrow \qquad \operatorname*{w-lim}_{n\to\infty} B_n A_n = BA.\text{”}$$

☐ true ☑ false

Question 3 (Dual Space)**.** Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider the Banach spaces $\ell^p(\mathbb{N})$, $\ell^q(\mathbb{N})$ and $\ell^p(\mathbb{N})^*$, where

$$\ell^p(\mathbb{N}) := \Big\{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_p := \Big(\sum_{j=1}^\infty |a_j|^p\Big)^{\frac{1}{p}} < \infty\Big\}.$$

Let $b = (b_j)_{j=1}^\infty \in \ell^q(\mathbb{N})$. Define

$$a_j = \begin{cases} \frac{|b_j|^q}{b_j} & \text{if } b_j \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) [5p] Show that $a = (a_j)_{j=1}^\infty \in \ell^p(\mathbb{N})$.
[HINT: $\frac{1}{p} + \frac{1}{q} = 1 \iff \frac{q+p}{pq} = 1 \iff \ldots$. Show first that $\|a\|_p^p = \|b\|_q^q$.]

Since $\frac{1}{p} + \frac{1}{q} = 1$, it follows that $q = p(q-1)$. Then $\|a\|_p^p = \sum |a_j|^p = \sum \left|\frac{|b_j|^q}{b_j}\right|^p = \sum |b_j|^{(q-1)p} = \sum |b_j|^q = \|b\|_q^q < \infty$ because $b \in \ell^q(\mathbb{N})$. So $a \in \ell^p(\mathbb{N})$.

(b) [5p] Show that $\|b\|_q^{q-1} = \|a\|_p$.

Note first that $q - 1 = \frac{q}{p}$. So $\|b\|_q^{q-1} = \|b\|_q^{\frac{q}{p}} = \left(\|b\|_q^q\right)^{\frac{1}{p}} = \left(\|a\|_p^p\right)^{\frac{1}{p}} = \|a\|_p$ by the proof of part (a).

For each $y \in \ell^q(\mathbb{N})$, define $l_y : \ell^p(\mathbb{N}) \to \mathbb{C}$ by

$$l_y(x) = \sum_{j=1}^\infty y_j x_j.$$

(c) [5p] Use the Hölder Inequality to show that $\|l_y\| \leq \|y\|_q$ for all $y \in \ell^q(\mathbb{N})$.

Let $y \in \ell^q(\mathbb{N})$. By the Hölder Inequality, $|l_y(x)| = |\sum y_j x_j| \leq \sum |y_j x_j| = \|yx\|_1 \leq \|y\|_q \|x\|_p$ for all $x \in \ell^p(\mathbb{N})$. Therefore $\|l_y\| \leq \|y\|_q$.

(d) [8p] Show that $\|l_y\| = \|y\|_q$ for all $y \in \ell^q(\mathbb{N})$.
[HINT: Choose $x \in \ell^p(\mathbb{N})$ such that $x_j y_j = |y_j|^q$. Why can we always do this? Use part (b).]

Let $y \in \ell^q(\mathbb{N})$. Choose $x \in \ell^p(\mathbb{N})$ such that $x_n y_n = |y_n|^q$. We can always do this by part (a). Then $|l_y(x)| = |\sum y_j x_j| = \sum |y_j|^q = \|y\|_q^q = \|y\|_q \|y\|_q^{q-1} = \|y\|_q \|x\|_p$ by part (b). It follows that $\|l_y\| = \sup_{\|x\|_p = 1} |l_y(x)| = \|y\|_q$.

(e) [2p] Show that $l_y \in \ell^p(\mathbb{N})^*$ for all $y \in \ell^q(\mathbb{N})$.

We showed in part (c) that l_y is bounded. It is easy to show that l_y is linear. Therefore $l_y \in \ell^p(\mathbb{N})^*$ for all $y \in \ell^q(\mathbb{N})$.

Question 4 (Closed Operators)**.** Let X and Y be a Banach spaces.

(a) [4p] Give the definition of the *graph* of an operator $A : \mathfrak{D}(A) \subseteq X \to Y$.

$$\Gamma(A) = \{(x, Ax) : x \in \mathfrak{D}(A)\} \subseteq X \oplus Y.$$

(b) [4p] Give the definition of a *closed operator*.

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closed iff, its graph is a closed subset of $X \oplus Y$.

(c) [8p] Now let $A : X \to Y$ be an operator. Suppose that A satisfies the following property:

- Let (x_n) be any sequence in X. If $x_n \to x$ and $Ax_n \to y$, then $Ax = y$.

Show that A is a closed operator.

[HINT: Start by letting (x_n, Ax_n) be any Cauchy sequence in $\Gamma(A)$.]

Let(x_n, Ax_n) be a Cauchy sequence in $\Gamma(A)$. Then x_n is a Cauchy sequence in X and Ax_n is a Cauchy sequence in Y. So $x_n \to x \in X$ and $Ax_n \to y \in Y$.

By the above property, $y = Ax$. So $(x_n, Ax_n) \to (x, Ax) \in \Gamma(A)$. Hence $\Gamma(A)$ is a closed set. Therefore A is a closed operator.

(d) [8p] Now let X be a Hilbert space. Let $A : X \to X$ be a symmetrical operator [i.e. $\langle x, Ay\rangle = \langle Ax, y\rangle\ \forall x, y \in X$]. Let (x_n) be a sequence such that $x_n \to x \in X$ and $Ax_n \to y \in X$.
Show that $Ax = y$.

First,

$$\begin{aligned}\langle z, y\rangle &= \left\langle z, \lim_{n\to\infty} Ax_n \right\rangle \\ &= \lim_{n\to\infty} \langle z, Ax_n\rangle \\ &= \lim_{n\to\infty} \langle Az, x_n\rangle \\ &= \left\langle Az, \lim_{n\to\infty} x_n \right\rangle \\ &= \langle Az, x\rangle \qquad = \langle z, Ax\rangle\end{aligned}$$

for all $z \in X$. Therefore $Ax = y$.

(e) [1p] Is the following statement true or false?

"Every symmetrical operator, defined on a Hilbert space, is a closed operator."

☑ true ☐ false

Question 5 (Weak Convergence)**.**

(a) [5p] Let X be a Banach space. Give the definition of *weak convergence* in X [i.e. $x_n \rightharpoonup x$ for $x_n \in X$.].

We say that x_n converges weakly to x, and write $x_n \rightharpoonup x$, iff $l(x_n) \to l(x)$ for all $l \in X^*$.

Now let X be a Hilbert space. Let $\{u_j\}_{j=1}^{\infty} \subseteq X$ be a countable, infinite, orthonormal set.

(b) [10p] Show that

$$\langle g, u_n\rangle \to 0$$

as $n \to \infty$, for all $g \in X$.

Let $g \in X$. First recall Bessel's Inequality: $\sum_{j=1}^{n} |\langle f, u_j\rangle|^2 \leq \|f\|$. It follows from Bessel's Inequality that $\sum_{j=1}^{\infty} |\langle g, u_j\rangle|^2$ is convergent, and hence that

$$\langle g, u_n\rangle \to 0$$

as $n \to \infty$, by the Divergence Theorem from 2^{nd}-year Calculus.

(c) [5p] Show that $u_n \rightharpoonup 0$ as $n \to \infty$.

Follows immediately from the Reisz Lemma and part (b).

(d) [5p] Show that $u_n \not\to 0$ as $n \to \infty$.

Since the u_j are orthonormal, we must have $\|u_j\| = 1$ for all j. Therefore $u_j \not\to 0$ as $n \to \infty$.