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MATEMATİK BÖLÜMÜ

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Question 1 (Weak Convergence)**.** Let X be a Banach space.

(a) [10p] Let (x_n) be a sequence in X. Give the definition of x_n *converges weakly* to x (i.e. $x_n \rightharpoonup x$ as $n \to \infty$).

> We say that x_n *converges weakly* to x (and write $x_n \rightharpoonup x$ as $n \to \infty$) iff $l(x_n) \to l(x)$ as $n \to \infty$, for all $l \in X^*$.

(b) [20p] Show that the weak limit is unique (i.e. show that if $x_n \rightharpoonup x$ and $x_n \rightharpoonup \tilde{x}$, then $x = \tilde{x}$).

> Suppose that $x_n \rightharpoonup x$ and $x_n \rightharpoonup \tilde{x}$. Then
>
> $$l(\tilde{x} - x) = l(\tilde{x}) - l(x) = \lim_{n\to\infty} l(x_n) - \lim_{n\to\infty} l(x_n) = 0$$
>
> for all $l \in X^*$. It follows that $\tilde{x} - x = 0$. Hence weak limits are unique.

(c) [20p] Suppose that $x_n \rightharpoonup x$. Let $A \in \mathcal{B}(X)$. Show that $Ax_n \rightharpoonup Ax$.

> Since $x_n \rightharpoonup x$, we know that $\tilde{l}(x_n) \to \tilde{l}(x)$ for all $\tilde{l} \in X^*$.
>
> Now let $l \in X^* = \mathcal{B}(X, \mathbb{C})$. We know that the composition of 2 bounded operators is bounded, so $l \circ A \in \mathcal{B}(X, \mathbb{C}) = X^*$.
>
> Therefore $lAx_n \to lAx$ for all $l \in X^*$. If follows that $Ax_n \rightharpoonup Ax$.

Question 2 (Reflexive Spaces)**.** Let X be a Banach space, with dual space X^* and double dual space X^{**}. Define the map $J : X \to X^{**}$ by

$$J(x)(l) = l(x)$$

for all $l \in X^*$.

(a) [15p] Show that $\|J(x)\|_{X^{**}} \leq \|x\|_X$ for all $x \in X$.

> Since $|J(x)(l)| = |l(x)| \leq \|l\|_{X^*} \|x\|_X$ for all $l \in X^*$, it follows that $\|J(x)\|_{X^{**}} \leq \|x\|_X$.

(b) [10p] Give the definition of a *reflexive* space.

> The space X is called reflexive iff $J(X) = X^{**}$.

In class we proved that:

- X is reflexive $\implies$ X^* is reflexive;

and

- If X is reflexive, and $Y \subseteq X$ is a closed subspace, then Y is reflexive.

(c) [25p] Show that

$$X^* \text{ is reflexive} \qquad \implies \qquad X \text{ is reflexive.}$$

[HINT: $X \cong J(X)$]

Suppose that X^* is reflexive. Then we know that X^{**} is reflexive. Since $J(X) \subseteq X^{**}$ is a closed subspace of X^{**}, it follows that $J(X)$ is also reflexive. Finally, since $J(X)$ is isomorphic to X we are finished.

Question 3 (The Hahn-Banach Theorem). First recall the real version of the Hahn-Banach Theorem:

The Hahn-Banach Theorem (Real version). *Let X be a real vector space, let $Y \subseteq X$ be a subspace, and let $\phi : X \to \mathbb{R}$ be a convex function. Suppose that $l : Y \to \mathbb{R}$ is a linear functional which satisfies*

$$l(y) \leq \phi(y) \qquad \forall y \in Y.$$

Then $\exists$ an extension $\bar{l} : X \to \mathbb{R}$ which satisfies

$$\bar{l}(x) \leq \phi(x) \qquad \forall x \in X.$$

Now let X be a complex vector space, let $Y \subseteq X$ be a subspace, and let $\phi : X \to \mathbb{R}$ be a convex function satisfying

$$\phi(\alpha x) \leq \phi(x) \qquad \forall\, |\alpha| = 1.$$

Suppose that $l : Y \to \mathbb{R}$ is a linear functional which satisfies

$$|l(y)| \leq \phi(y) \qquad \forall y \in Y.$$

(a) [10p] Let $l_r = \mathrm{Re}(l)$. Show that

$$l(x) = l_r(x) - i l_r(ix).$$

$l(x) = \mathrm{Re}(l(x)) + i\,\mathrm{Im}(l(x)) = \mathrm{Re}(l(x)) + i\,\mathrm{Re}(-il(x)) = l_r(x) - il_r(ix)$.

(b) [10p] Show that l_r has a real linear extension $\bar{l}_r : X \to \mathbb{R}$ satisfying $\bar{l}_r(x) \leq \phi(x)$ $\forall x \in X$.

Follows immediately from the real version of the Hahn-Banach Theorem.

Define $\bar{l}(x) := \bar{l}_r(x) - i\bar{l}_r(ix)$.

(c) [10p] Show that $\bar{l}$ is real linear. In other words, show that $\bar{l}(x+\lambda y) = \bar{l}(x)+\lambda\bar{l}(y)$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$.

Follows since $\bar{l}_r$ is real-linear.

(d) [10p] Show that $\bar{l}$ is complex linear. In other words, show that $\bar{l}(x + \lambda y) = \bar{l}(x) + \lambda\bar{l}(y)$ for all $x, y \in X$ and $\lambda \in \mathbb{C}$.
[HINT: First show that $\bar{l}(ix) = i\bar{l}(x)$.]

Since

$$\bar{l}(ix) = \bar{l}_r(ix) + i\bar{l}_r(x) = i\bar{l}(x),$$

this follows by part (c).

(e) [10p] Show that $\left|\bar{l}(x)\right| \leq \phi(x)$ for all $x \in X$.
[HINT: Use $\alpha = \frac{\overline{\bar{l}(x)}}{|\bar{l}(x)|}$ (where $\overline{a+ib}$ means $a - ib$), and $|\bar{l}(x)| = \alpha\bar{l}(x)$.]

Let $x \in X$. Defining α as in the hint, we see that $|\alpha| = 1$. Thus

$$\left|\bar{l}(x)\right| = \alpha\bar{l}(x) = \bar{l}(\alpha x) = \bar{l}_r(x) \leq \phi(\alpha x) \leq \phi(x).$$