2014.05.29 **MAT462 Fonksiyonel Analiz II – Final Sınavın Çözümleri** N. Course

Soru 1 (Finite Rank Operators)**.** Let X be a Hilbert space.

(a) [5p] Give the definition of a *finite rank operator* $K \in \mathcal{B}(X)$.

> An operator $K \in \mathcal{B}(X)$ is called a *finite rank operator* iff $\mathrm{Ran}(K)$ is finite dimensional.

(b) [10p] Show that

$$A \in \mathcal{B}(X) \text{ is a finite rank operator } \implies A \text{ is compact.}$$

[HINT: Use the Heine-Borel Theorem.]

> Heine-Borel says that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
>
> Let f_n be a bounded sequence. Then $\{Af_n\}$ is bounded. Since $\mathrm{Ran}(A)$ is finite dimensional, it follows that $\{Af_n\}$ is contained in a compact subset $Y \subseteq X$. Y is sequentially compact, hence there exists a convergent subsequence. Therefore A is compact.

Define

$$\Omega := \{A \in \mathcal{B}(X) : A \text{ is a finite rank operator}\}.$$

(c) [10p] Show that

$$\overline{\Omega} \subseteq \mathcal{K}(X).$$

[HINT: $\overline{\Omega}$ denotes the closure of Ω.]

> Suppose $K_n \in \Omega$ and $K_n \to K$. Then $K_n \in \mathcal{K}(X)$ by part (b). Since the limit of a sequence of compact operators is compact, we have that $K \in \mathcal{K}(X)$.

Soru 2 (Weak and Strong Convergence of Operators)**.** Consider the Hilbert space $\ell^2(\mathbb{N}) = \big\{a = (a_j)_{j=1}^\infty \subseteq \mathbb{C} : \|a\|_2 < \infty\big\}$ with the inner product $\langle x, y\rangle_2 = \sum_{j=1}^\infty \overline{x_j} y_j$.

Define a sequence of (bounded linear) operators $S_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S_n(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \ldots).$$

Let the operator $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$K = \left\langle \delta^1, \cdot \right\rangle_2 \delta^1$$

where δ^1 is the sequence $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)$.

(a) [5p] Show that $\|S_n\| = 1, \ \forall n \in \mathbb{N}$.

> First $\|S_n x\|_2 = (\sum_{j=1}^\infty |(S_n x)_j|^2)^{\frac{1}{2}} = (\sum_{j=n+1}^\infty |x_j|^2)^{\frac{1}{2}} \leq (\sum_{j=1}^\infty |x_j|^2)^{\frac{1}{2}} = \|x\|_2$ for all x. So $\|S_n\| \leq 1$. Moreover, $\|\delta^m\|_2 = 1$ for all $m \in \mathbb{N}$, and $\left\|S_n \delta^{n+1}\right\|_2 = \left\|\delta^1\right\|_2 = 1 = \left\|\delta^{n+1}\right\|_2$. Therefore $\|S_n\| \geq 1$.

(b) [3p] Show that $S_n \not\to 0$ as $n \to \infty$.

Since $\|S_n\| = 1$ for all n, it follows that $\|S_n - 0\| \not\to 0$. So $S_n \not\to 0$.

(c) [7p] Show that $\text{s-lim}_{n\to\infty} S_n = 0$.

Let $x \in \ell^2(\mathbb{N})$. Then $\|x\|_2 < \infty$. So $\sum_{j=1}^{n} |x_j|^2 \to \sum_{j=1}^{\infty} |x_j|^2$ as $n \to \infty$. Therefore

$$\|S_n x\|_2 = (\sum_{j=1}^{\infty} |(S_n x)_j|^2)^{\frac{1}{2}} = (\sum_{j=n+1}^{\infty} |x_j|^2)^{\frac{1}{2}} \to 0$$

as $n \to \infty$. Hence $S_n x \to 0$ for all x, and thus $\text{s-lim}_{n\to\infty} S_n = 0$.

(d) [10p] Show that $S_n K \to 0$, but $K S_n \not\to 0$.

Since $\|K S_n x\|_2 = \|(x_{n+1}, 0, 0, 0, 0, 0, \ldots)\|_2 = |x_{n+1}|$, it follows that $\|K S_n\| = 1$ for all n. Therefore $K S_n \not\to 0$.

Finally, since $S_n K = 0$ for all n, it is obvious that $S_n K \to 0$.

Soru 3 (Hilbert-Schmidt Operators)**.** Let X be a Hilbert space.

(a) [5p] Give the definition of the *Hilbert-Schmidt norm*, $\|\cdot\|_2$.
[HINT: I do NOT want the ℓ^2-norm of a sequence (also called $\|\cdot\|_2$)!!! I want the Hilbert-Schmidt norm of an operator.]

$$\|K\|_2 := \left(\sum_j s_j(K)^2 \right)^{\frac{1}{2}}$$

where $\{s_j(K)\}$ are the singular values of $K : X \to X$.

(b) [5p] Give the definition of $\mathcal{J}_2(X)$, the space of *Hilbert-Schmidt operators*.

$$\mathcal{J}_2(X) := \{K \in \mathcal{K}(X) : \|K\|_2 < \infty\}.$$

Let $K \in \mathcal{J}_2(X)$ and let $A \in \mathcal{B}(X)$.

(c) [10p] Show that

$$\|AK\|_2 \leq \|A\| \, \|K\|_2 \, .$$

[HINT: $\|\cdot\|$ denotes the operator norm, and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.]

Let $\{u_j\}$ be an orthonormal basis of X. Then

$$\|AK\|_2^2 = \sum_j \|AKu_j\|^2 \leq \sum_j \|A\|^2 \|Ku_j\|^2 = \|A\|^2 \sum_j \|Ku_j\|^2 = \|A\|^2 \|K\|_2^2$$

by a Lemma[1] from the course.

[1] actually it is Lemma 5.5 , but you are not expected to remember numbers.

(d) [5p] Show that

$$\|KA\|_2 \leq \|K\|_2 \|A\| .$$

[HINT: $(BC)^* = C^*B^*$.]

First note that KA is compact because K is compact and A is bounded. Since $s_j(T) = s_j(T^*)$ for all compact operators T, it follows immediately that

$$\|KA\|_2 = \|(KA)^*\|_2 = \|A^*K^*\|_2 \leq \|A^*\| \|K^*\|_2 = \|A\| \|K\|_2 .$$

Soru 4 (Reflexive Spaces)**.** Let X be a Banach space, with dual space X^* and double dual space X^{**}. Define the map $J : X \to X^{**}$ by

$$J(x)(l) = l(x)$$

for all $l \in X^*$.

(a) [8p] Show that $\|J(x)\|_{X^{**}} \leq \|x\|_X$ for all $x \in X$.

Since $|J(x)(l)| = |l(x)| \leq \|l\|_{X^*} \|x\|_X$ for all $l \in X^*$, it follows that $\|J(x)\|_{X^{**}} \leq \|x\|_X$.

(b) [5p] Give the definition of a *reflexive* space.

The space X is called reflexive iff $J(X) = X^{**}$.

In class we proved that:

- X is reflexive $\implies$ X^* is reflexive;

and

- If X is reflexive, and $Y \subseteq X$ is a closed subspace, then Y is reflexive.

(c) [12p] Show that

$$X^* \text{ is reflexive} \qquad \implies \qquad X \text{ is reflexive.}$$

[HINT: $X \cong J(X)$]

Suppose that X^* is reflexive. Then we know that X^{**} is reflexive. Since $J(X) \subseteq X^{**}$ is a closed subspace of X^{**}, it follows that $J(X)$ is also reflexive. Finally, since $J(X)$ is isomorphic to X we are finished.

Soru 5 (Weak Convergence)**.**

(a) [5p] Let X be a Banach space. Give the definition of *weak convergence* in X [i.e. $x_n \rightharpoonup x$ for $x_n \in X$.].

We say that x_n converges weakly to x, and write $x_n \rightharpoonup x$, iff $l(x_n) \to l(x)$ for all $l \in X^*$.

Consider the Banach space $\ell^p(\mathbb{N})$ where

$$\ell^p(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \|a\|_p := \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$, and

$$\ell^\infty(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \|a\|_\infty := \sup_j |a_j| < \infty \right\}.$$

Define

$$\delta_j^n = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j. \end{cases}$$

[For example, δ^5 is the sequence $(0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,\dots)$]

(b) [6p] Show that $\delta^n \in \ell^p(\mathbb{N})$ for all $n \in \mathbb{N}$ and for all $1 \leq p \leq \infty$.

Clearly

$$\|\delta^n\|_\infty = \sup_j \left|\delta_j^n\right| = |\delta_n^n| = 1 < \infty.$$

So $\delta^n \in \ell^\infty(\mathbb{N})$ for all n.

Let $1 \leq p < \infty$. Then

$$\|\delta^n\|_p^p = \sum_{j=1}^{\infty} \left|\delta_j^n\right|^p = \|\delta_n^n\| = 1 < \infty.$$

So $\delta^n \in \ell^p(\mathbb{N})$ for all n.

(c) [7p] Let $1 < p < \infty$. Show that $\delta^n \rightharpoonup 0$.

Let $l \in \ell^p(\mathbb{N})^*$. Then $\exists\, y \in \ell^q(\mathbb{N})$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that

$$l(x) = \sum_{j=1}^{\infty} y_j x_j$$

for all $x \in \ell^p(\mathbb{N})$.
So

$$|l(\delta^n)| = \left| \sum_{j=1}^{\infty} y_j \delta_j^n \right| = |y_n| \to 0$$

as $n \to \infty$ since $y \in \ell^q(\mathbb{N})$.
Therefore $\delta \rightharpoonup 0$ as $n \to \infty$.

(d) [7p] Show that δ^n is not weakly convergent in $\ell^1(\mathbb{N})$.

Consider first the functional $l \in \ell^1(\mathbb{N})^*$ defined by $l(x) = x_1$. Then $l(\delta^n) = 0$ for all $n \geq 2$. So clearly $l(\delta^n) \to 0 = l(0)$. Therefore, *if* δ^n is weakly convergent in $\ell^1(\mathbb{N})$, then $\delta^n \rightharpoonup 0$. Next consider $\tilde{l} \in \ell^1(\mathbb{N})^*$ defined by $\tilde{l}(x) = x_1 + x_2 + x_3 + x_4 + \dots$. Then $\tilde{l}(\delta^n) = 1$ for all $n \in \mathbb{N}$, so $\tilde{l}(\delta^n) \not\to 0$ as $n \to \infty$.
Therefore δ^n is not weakly convergent in $\ell^1(\mathbb{N})$.