2014.04.10 **MAT462 Fonksiyonel Analiz II – Ara Sınavın Çözümleri** N. Course

Soru 1 (Closed Operators)**.** Let X and Y be Banach spaces.

(a) [8p] Give the definition of the *graph* of an operator $A : \mathfrak{D}(A) \subseteq X \to Y$.

$$\Gamma(A) = \{(x, Ax) : x \in \mathfrak{D}(A)\} \subseteq X \oplus Y.$$

(b) [7p] Give the definition of a *closed operator*.

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closed iff, its graph is a closed subset of $X \oplus Y$.

Consider the operator $B : \ell^2(\mathbb{N}) \to \mathrm{Ran}(B)$ given by

$$B(a_1, a_2, a_3, a_4, \ldots, a_j, \ldots) = \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots, \frac{a_j}{j}, \ldots\right)$$

and its inverse $B^{-1} : \mathrm{Ran}(B) \to \ell^2(\mathbb{N})$ given by

$$B^{-1}(a_1, a_2, a_3, a_4, \ldots, a_j, \ldots) = (a_1, 2a_2, 3a_3, 4a_4, \ldots, ja_j, \ldots).$$

(c) [10p] Show that

$$\mathrm{Ran}(B) \neq \ell^2(\mathbb{N}).$$

[HINT: Is $b = \left(\frac{1}{j}\right)_{j=1}^{\infty}$ in $\ell^2(\mathbb{N})$? Is b in $\mathrm{Ran}(B)$?]

As per the hint, we consider the sequence $b = \left(\frac{1}{j}\right)_{j=1}^{\infty}$. Since

$$\|b\|_2 = \sqrt{\sum_{j=1}^{\infty} |b_j|^2} = \sqrt{\sum_{j=1}^{\infty} \frac{1}{j^2}} < \infty$$

(see MAT234), it follows that $b \in \ell^2(\mathbb{N})$.

However, if there exists $a \in \ell^2(\mathbb{N})$ such that $Ba = b$, then we get the contradiction

$$\infty = \sum_{j=1}^{\infty} 1 = \sum_{j=1}^{\infty} |jb_j|^2 = \sum_{j=1}^{\infty} |a_j|^2 < \infty.$$

Therefore $b \notin \mathrm{Ran}(B)$ and hence $\mathrm{Ran}(B) \neq \ell^2(\mathbb{N})$.

Now suppose that

- $A : \mathfrak{D}(A) \to \mathrm{Ran}(A)$ is a closed operator;
- $\mathfrak{D}(A) \subseteq X$;
- $\mathrm{Ran}(A) \subseteq Y$;
- A is injective ($x \neq y \implies Ax \neq Ay$); and
- $A^{-1} : \mathrm{Ran}(A) \to \mathfrak{D}(A)$ is the inverse of A.

(d) [25p] Show that A^{-1} is a closed operator.

Let $(y_n, A^{-1}y_n)$ be a Cauchy sequence in $\Gamma(A^{-1})$. Then y_n is a Cauchy sequence in $\operatorname{Ran}(A)$ and $A^{-1}y_n$ is a Cauchy sequence in $\mathfrak{D}(A)$

Because $y_n \in \operatorname{Ran}(A)$ and because A is injective, $\exists$ a unique $x_n \in \mathfrak{D}(A)$ such that $y_n = Ax_n$. It follows that x_n and Ax_n are Cauchy sequences, and hence that (x_n, Ax_n) is a Cauchy sequence in $\Gamma(A)$.

Since X and Y are Banach spaces, we know that $x_n \to x$ and $Ax_n \to y$, for some $x \in X$ and $y \in Y$. Then since $\Gamma(A)$ is closed, we know that $y = Ax$ and that $(x_n, Ax_n) \to (x, Ax)$. It follows that $(y_n, A^{-1}y_n) = (Ax_n, x_n) \to (Ax, x) = (y, A^{-1}y) \in \Gamma(A^{-1})$ and hence that A^{-1} is a closed operator.

Soru 2 (Weak Convergence)**.** Let X be a Banach space.

(a) [10p] Let (x_n) be a sequence in X. Give the definition of x_n *converges weakly* to x (i.e. $x_n \rightharpoonup x$ as $n \to \infty$).

We say that x_n *converges weakly* to x (and write $x_n \rightharpoonup x$ as $n \to \infty$) iff $l(x_n) \to l(x)$ as $n \to \infty$, for all $l \in X^*$.

(b) [10p] Let (x_n) be a sequence in X. Show that

$$x_n \to x \text{ as } n \to \infty \qquad \Longrightarrow \qquad x_n \rightharpoonup x \text{ as } n \to \infty$$

Suppose that $x_n \to x$. Let $l \in X^*$. Then $\|l\| < \infty$. It follows that

$$|l(x_n) - l(x)| = |l(x_n - x)| \leq \|l\| \, \|x_n - x\| \to 0,$$

and hence $l(x_n) \to l(x)$. Therefore $x_n \rightharpoonup x$.

Now let X be a Hilbert space and let (f_n) be a sequence in X. Suppose that $f_n \rightharpoonup f$ as $n \to \infty$.

(c) [20p] Show that

$$f_n \to f \text{ as } n \to \infty \qquad \Longleftarrow \qquad \limsup_{n\to\infty} \|f_n\| \leq \|f\|$$

[HINT: We proved in class that $f_n \rightharpoonup f \implies \|f\| \leq \liminf_{n\to\infty} \|f_n\|$.]
[HINT: First, try to show that $\|f_n\| \to \|f\|$. Then use this to prove that $\|f_n - f\| \to 0$.]

Using the hints, this should be quite an easy question:

Since

$$\|f\| \leq \liminf_{n\to\infty} \|f_n\| \leq \limsup_{n\to\infty} \|f_n\| \leq \|f\|$$

it follows that $\lim_{n\to\infty} \|f_n\|$ exists and

$$\lim_{n\to\infty} \|f_n\| = \|f\| \, .$$

Moreover, since $f_n \rightharpoonup f$, we have that

$$\langle g, f_n \rangle \to \langle g, f \rangle$$

for all g. Therefore

$$\|f - f_n\|^2 = \|f\|^2 - 2 \operatorname{Re} \langle f, f_n \rangle + \|f_n\|^2 \to \|f\|^2 - 2 \operatorname{Re} \langle f, f \rangle + \|f\|^2 = 0$$

and hence $f_n \to f$.

(d) [10p] Show that

$$f_n \to f \text{ as } n \to \infty \qquad \Longrightarrow \qquad \limsup_{n\to\infty} \|f_n\| \leq \|f\|$$

Since $f_n \to f$, we know that $\lim_{n\to\infty} \|f_n\| = \|f\|$. Therefore

$$\limsup_{n\to\infty} \|f_n\| = \liminf_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \|f_n\| = \|f\|$$

and we are done.

Soru 3 (The Hahn-Banach Theorem)**.** Let X be a Banach space.

(a) [10p] Give the definition of a *convex function* $\phi : X \to \mathbb{R}$.

The function $\phi : X \to \mathbb{R}$ is called *convex* iff

$$\phi\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $\lambda \in (0,1)$.

(b) [20p] Let $Y \subseteq X$ be a subspace and let $l \in Y^*$. Show that $\exists\, \bar{l} \in X^*$ such that

(a) $l(y) = \bar{l}(y)$ for all $y \in Y$; and

(b) $\|l\| = \|\bar{l}\|$.

Using the convex function $\phi(x) = \|l\|\,\|x\|$, this follows by the Hahn-Banach Theorem. More details please.

(c) [20p] Let $x_1, \ldots, x_n \in X$ be linearly independent vectors and let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Show that $\exists\, l \in X^*$ such that $l(x_k) = \alpha_k$ for all $k = 1, \ldots, n$.

This was a homework question, so there is no excuse for getting less than full marks on this part:

Define $M = \operatorname{span}\{x_1, \ldots, x_n\}$ and define $l : M \to \mathbb{C}$ by $l(\sum_j \lambda_j x_j) = \sum_j \lambda_j \alpha_j$. Then use the Hahn-Banach Theorem to extend l to $\bar{l} : X \to \mathbb{C}$.