2015.05.21 **MAT462 Fonksiyonel Analiz II – Final Sınavın Çözümleri** N. Course

**Soru 1 (Weak Convergence and Weak-* Convergence of Functionals).** Let $X$ be a Banach space. Let $\{l_n\}_{n=1}^{\infty} \subseteq X^*$ be a sequence of linear functionals.

(a) [5p] Give the definition of $l_n \rightharpoonup l$.
[HINT: $(X^*)^* = X^{**}$.]

We say that $l_n$ converges weakly to $l$, and write $l_n \rightharpoonup l$, iff $j(l_n) \to j(l)$ for all $j \in X^{**}$.

(b) [5p] Give the definition of the *weak-* limit* of $l_n$.

We say that $l$ is the *weak-* limit* of $l_n$ if and only if $l_n(x) \to l(x)$ for all $x \in X$.

(c) [5p] Show that

$$l_n \to l \qquad \Longrightarrow \qquad l_n \rightharpoonup l.$$

Suppose $l_n \to l$. Then $\|l_n - l\| \to 0$. Let $j \in X^{**}$. Then $j$ is bounded. Therefore

$$\|j(l_n) - j(l)\| \leq \|j\| \, \|l_n - l\| \to 0$$

and hence $j(l_n) \to j(l)$. So $l_n \rightharpoonup l$.

(d) [10p] Show that

$$l_n \rightharpoonup l \qquad \Longrightarrow \qquad l \text{ is the weak-* limit of } l_n$$

[HINT: Consider the map $J : X \to X^{**}$ defined by $J(x)(l) = l(x)$.]

Since

$$\begin{aligned} l \text{ is the weak-* limit of } l_n &\iff l_n(x) \to l(x) \ \forall x \in X \\ &\iff j(l_n) \to j(l) \ \forall j \in J(X) \end{aligned}$$

the result follows immediately from $J(X) \subseteq X^{**}$.

**Soru 2 (Singular Values).** Let $X$ be a Hilbert space and let $K : X \to X$ be compact.

(a) [5p] Give the definition of the *singular values* of $K$.

For all $f \in X$, we can write

$$K^*Kf = \sum_j s_j^2 \langle u_j, f \rangle u_j$$

where the $u_j$ are orthonormal. The numbers $s_j \in \mathbb{R}$, $s_j > 0$ are called singular values of $K$.

Now let $X = \ell^2(\mathbb{N})$. Suppose that

- $(\mu_n)$ is a sequence;
- $\mu_n \in \mathbb{C}$ for all $n$;
- $\mu_n \neq 0$ for all $n$; and
- $\lim_{n\to\infty} \mu_n = 0$.

Define an operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$A\big(x_1, x_2, x_3, x_4, \dots\big) = \big(\mu_1 x_1, \mu_2 x_2, \mu_3 x_3, \mu_4 x_4, \dots\big).$$

(b) [5p] Calculate $A^*$ and $A^*A$.

Since

$$\langle A^* f, g\rangle = \langle f, Ag\rangle = \sum_j \overline{f}_j (Ag)_j = \sum_j \overline{f}_j \mu_j g_j = \sum_j \overline{\overline{\mu}_j f_j} g_j = \langle(\overline{\mu}_1 f_1, \overline{\mu}_2 f_2, \dots), g\rangle,$$

we can see that $A^*$ multiplies $f$ by the sequence $\overline{\mu}_n$. It is then easy to see that

$$A^*Ax = (\overline{\mu}_1\mu_1 x_1, \overline{\mu}_2\mu_2 x_2, \dots) = (|\mu_1|^2 x_1, |\mu_2|^2 x_2, \dots).$$

(c) [9p] Find the eigenvalues and orthonormal eigenvectors of $A^*A$.

If $x \neq 0$ and

$$\alpha x = A^*Ax = \big(|\mu_1|^2 x_1, |\mu_2|^2 x_2, |\mu_3|^2 x_3, |\mu_4|^2 x_4, \dots\big),$$

then clearly $\alpha = |\mu_j|^2$ for some $j$. Conversely, clearly

$$|\mu_j|^2 \delta^j = A^*A\delta^j.$$

So each $\alpha_n := |\mu_n|^2$ is an eigenvalue with corresponding eigenvector $u_n := \delta^n$. Clearly $\{\delta^n\}$ is an orthonormal set.

(Note: We might have repeating eigenvalues. For example, if $\mu_1 = 1$ and $\mu_7 = i$, then $\alpha_1 = \alpha_7$.) At this point, remove all repeats from the list $\alpha_1, \alpha_2, \alpha_3, \dots$ and throw away the corresponding vectors. This leaves subsequences $(\alpha_{n_j})_{j=1}^J$ and $(\delta^{n_j})_{j=1}^J$, where $J \in \mathbb{N} \cup \{\infty\}$.

(d) [6p] Find the singular values of $A$.

The singular values are given by

$$s_j = \|Au_{n_j}\| = |\mu_{n_j}|.$$

**Soru 3 (Hilbert-Schmidt Operators and Trace Class Operators).** Let $X$ be a Hilbert space.

(a) [3p] Give the definition of the *Hilbert-Schmidt norm*, $\|\cdot\|_2$.
[HINT: I do NOT want the $\ell^2$-norm of a sequence (also called $\|\cdot\|_2$)!!! I want the Hilbert-Schmidt norm of an operator.]

$$\|K\|_2 := \left(\sum_j s_j(K)^2\right)^{\frac{1}{2}}$$

where $\{s_j(K)\}$ are the singular values of $K : X \to X$.

(b) [3p] Give the definition of $\mathcal{J}_2(X)$, the space of *Hilbert-Schmidt operators*.

$$\mathcal{J}_2(X) := \{K \in \mathcal{K}(X) : \|K\|_2 < \infty\}.$$

Let $K \in \mathcal{J}_1(X) \subseteq \mathcal{K}(X)$. Then we know that

$$K = \sum_j s_j \langle u_j, \cdot\rangle v_j,$$

where $s_j = s_j(K)$ are the singular values of $K$, $u_j$ are orthonormal eigenvalues of $K^*K$ and $v_j := \frac{1}{s_j} K u_j$.

Define

$$K_1 := \sum_j \sqrt{s_j} \langle u_j, \cdot\rangle v_j \qquad \text{and} \qquad K_2 := \sum_j \sqrt{s_j} \langle u_j, \cdot\rangle u_j.$$

(c) [8p] Show that $K_1 K_2 = K$.

Since the $u_j$ are orthonormal, we have that

$$\begin{aligned} K_1 K_2 f &= \sum_j \sqrt{s_j} \langle u_j, K_2 f\rangle v_j \\ &= \sum_j \sqrt{s_j} \left\langle u_j, \sum_n \sqrt{s_n} \langle u_n, f\rangle u_n \right\rangle v_j \\ &= \sum_j \sum_n \sqrt{s_j}\sqrt{s_n} \langle u_n, f\rangle \langle u_j, u_n\rangle v_j \\ &= \sum_j s_j \langle u_j, f\rangle v_j \\ &= K f \end{aligned}$$

for all $f \in X$.

(d) [11p] Show that $K_1, K_2 \in \mathcal{J}_2(X)$.

We proved in class that $\|A\|_2 < \infty \implies A$ is compact – so we only need to prove that the 2-norms of $K_1$ and $K_2$ are finite.

By Lemma 5.5,

$$\begin{aligned} \|K_1\|_2^2 = \sum_k \|K_1 u_k\|^2 = \sum_k \left\| \sum_j \sqrt{s_j} \langle u_j, u_k\rangle v_j \right\|^2 = \sum_k \|\sqrt{s_k} v_k\|^2 \\ = \sum_k s_k = \|K\|_1 < \infty \end{aligned}$$

since the $u_j$ are orthonormal and since $\|v_k\| = 1$ and $s_k > 0$ for all $k$.

A similar argument shows that $\|K_2\|_2 < \infty$. Therefore $K_1, K_2 \in \mathcal{J}_2(X)$.

**Soru 4 (Closed Operators).** Let $X$ and $Y$ be normed spaces.

(a) [5p] Give the definition of a *closed* operator $A : \mathfrak{D}(A) \subseteq X \to Y$.

An operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is called a closed operator if the graph of $A$ is a closed subset of $X \oplus Y$.

Now let $X = Y = \ell^2(\mathbb{N})$. Consider the linear operator $T : \mathfrak{D}(T) \to \ell^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, x_4, \dots) := (x_1, 2x_2, 3x_3, 4x_4, \dots)$$

where

$$\mathfrak{D}(T) := \{x \in \ell^2(\mathbb{N}) : \exists N \in \mathbb{N} \text{ such that } x_n = 0 \ \forall n > N\}.$$

(b) [6p] Is $T$ bounded? [Prove your answer]

$T$ is not bounded.

Clearly $\|\delta^n\|_2 = 1$ and $\delta^n \in \mathfrak{D}(T)$ for all $n$, but $\|T\delta^n\|_2 = n \to \infty$.

(c) [14p] Is $T$ closed? [Prove your answer]

[HINT: Consider the sequence of sequences $(a^n)$ defined by $a_j^n := \begin{cases} \frac{1}{j^2} & j \leq n \\ 0 & j > n \end{cases}$.]

$T$ is not closed.

Define $a_j^n := \begin{cases} \frac{1}{j^2} & j \leq n \\ 0 & j > n \end{cases}$. Clearly $a^n \in \mathfrak{D}(T)$ for all $n$. Moreover, $a^n \to (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots) =: a$ and $a \in \ell^2(\mathbb{N})$ because $\|a\|_2^2 = \sum_j \frac{1}{j^4} < \infty$, but $a \notin \mathfrak{D}(T)$.

Note that $Ta^n = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n^2}, 0, 0, 0, 0, \dots) \to (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) =: b$. Since $\|b\|_2^2 = \sum_j \frac{1}{n^2} < \infty$, $b \in \ell^2(\mathbb{N})$ also.

Since $(a^n, Ta^n) \in \Gamma(T)$ for all $n$, since $(a^n, Ta^n) \to (a, b) \in \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ and since $(a, b) \notin \Gamma(T)$, we have that $T$ is not closed.

**Soru 5 (Weak Cauchy Seqences).** Let $X$ be a Banach space.

(a) [5p] Give the definition of a *weak Cauchy sequence* in $X$.

> A sequence $x^n$ is called *weak Cauchy* iff $l(x^n)$ is Cauchy for all $l \in X^*$.

Now let $c_0(\mathbb{N}) := \{x = (x_j)_{j=1}^{\infty} \subseteq \mathbb{C} : x_j \to 0\} \subseteq \ell^\infty(\mathbb{N})$ and $\|x\|_\infty := \sup_j |x_j|$. Then $(c_0(\mathbb{N}), \|\cdot\|_\infty)$ is a Banach space.

We have seen that its dual space, $c_0(\mathbb{N})^*$, is isomorphic to $\ell^1(\mathbb{N})$: This implies that $\forall\ l \in c_0(\mathbb{N})^*$, $\exists\ b \in \ell^1(\mathbb{N})$ such that $l(x) = \sum_j b_j x_j$ for all $x \in c_0(\mathbb{N})$.

Now let $a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C}$ be a bounded sequence and define a sequence of sequences $(x^n)$ by

$$
\begin{aligned}
x^1 &= (a_1, 0, 0, 0, 0, 0, 0, \dots) \\
x^2 &= (a_1, a_2, 0, 0, 0, 0, 0, \dots) \\
x^3 &= (a_1, a_2, a_3, 0, 0, 0, 0, \dots) \\
x^4 &= (a_1, a_2, a_3, a_4, 0, 0, 0, \dots) \\
&\ \vdots \\
x^n &= (a_1, \dots, a_n, 0, 0, 0, \dots) \\
&\ \vdots
\end{aligned}
$$

Clearly $x^n \in c_0(\mathbb{N})$ for all $n$.

(b) [10p] Show that $(x^n)$ is a weak Cauchy sequence in $c_0(\mathbb{N})$.
[HINT: $c_0(\mathbb{N})^* \cong \ell^1(\mathbb{N})$]

> Let $l \in c_0(\mathbb{N})$ and let $b \in \ell^1(\mathbb{N})$ be chosen such that $l(x) = \sum_j b_j x_j$ for all $x$.
>
> Now, if $n > m > N$ we have that
>
> $$
> \begin{aligned}
> |l(x^n) - l(x^m)| &= \left| \sum_{j=1}^{n} b_j a_j - \sum_{j=1}^{m} b_j a_j \right| = \left| \sum_{j=m+1}^{n} b_j a_j \right| \\
> &\leq \sum_{j=m+1}^{n} |b_j a_j| \leq \sum_{j=N}^{\infty} |b_j|\, |a_j| \to 0
> \end{aligned}
> $$
>
> as $N \to \infty$ because $a$ is bounded and $b \in \ell^1(\mathbb{N})$. Therefore $(x^n)$ is weak Cauchy.

(c) [10p] Is $(x^n)$ weakly convergent in $c_0(\mathbb{N})$? [Prove your answer.]

> For most choices of $a$, $(x^n)$ is not weakly convergent.
>
> For example, let $a = (1, 1, 1, 1, 1, 1, 1, 1, 1, \dots)$. Suppose that $x^n \rightharpoonup x$. Then clearly
>
> $$\sum_{j=1}^{\infty} b_j x_j = l(x) = \lim_{n\to\infty} l(x^n) = \lim_{n\to\infty} \sum_{j=1}^{n} b_j = \sum_{j=1}^{\infty} b_j$$
>
> for all $b \in \ell^1(\mathbb{N})$. Choosing $b = \delta^m$, we see that $x_m = 1$ for all $m$. But $(1, 1, 1, 1, 1, 1, 1, 1, 1, \dots) \notin c_0(\mathbb{N})$. Therefore $(x^n)$ is not weakly convergent in $c_0(\mathbb{N})$.