2015.03.26 **MAT462 Fonksiyonel Analiz II – Ara Sınavın Çözümleri** N. Course

Soru 1 (Closable Operators)**.**

(a) [7p] Give the definition of a *closed* operator.

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closed iff, its graph is a closed subset of $X \oplus Y$.

(b) [8p] Give the definition of a *closable* operator.

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closable iff,

$$\overline{\Gamma(A)} \cap \{(0, y) : y \in Y\} = \{(0, 0\}.$$

– or equivalently –

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closable iff $\exists$ an operator $\overline{A}$ such that $\overline{\Gamma(A)} = \Gamma(\overline{A})$.

Now suppose that $X = Y = \ell^2(\mathbb{N})$. Define an operator $B : \mathfrak{D}(B) \subseteq \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$Ba = \left(\sum_{j \in \mathbb{N}} a_j \right) \delta^1$$

where $\mathfrak{D}(B) = \ell^1(\mathbb{N})$. Here, δ^1 denotes the sequence $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)$.
Define (b^n) by

$$b_j^n = \begin{cases} \frac{1}{n} & 1 \leq j \leq n \\ 0 & j > n. \end{cases}$$

So for example, $b^6 = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, \ldots)$.

(c) [5p] Calculate $\|b^n\|_1$.

$\|b^n\|_1 = 1$

(d) [5p] Calculate $\|b^n\|_2$.

$\|b^n\|_2 = \frac{1}{\sqrt{n}}$

(e) [5p] Calculate Bb^n.

$Bb^n = \delta^1$

(f) [20p] Show that B is not closable.

Clearly $b^n \to 0$ (by part (d)), but $Bb^n = \delta^1 \not\to 0 = B0$. By a result from the course, B is not closable.

Soru 2 (The Hahn-Banach Theorem)**.** Let X be a normed space.

(a) [10p] Give the definition of a *convex* function on X.

A function $\phi : X \to \mathbb{R}$ is called *convex* iff

$$\phi\left(\lambda x + (1-\lambda)y\right) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in X$ and for all $\lambda \in (0,1)$.

Define

$$M := \{m = (m_j)_{j=1}^{\infty} \subseteq \mathbb{C} : \exists N \in \mathbb{N} \text{ such that } m_j = 0 \ \forall j > N\}.$$

(b) [5p] Show that M is a subspace of $\ell^\infty(\mathbb{N})$.

Clearly $M \subseteq \ell^\infty(\mathbb{N})$. It is easy to prove that if $x, y \in M$ and $\lambda \in \mathbb{C}$, then $x + \lambda y \in M$; which proves that M is a linear space.

(c) [5p] Show that M is not closed.

Let $x^n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots)$. Clearly $x^n \in M$ for all n. Moreover, it is easy to see that $x^n \to (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \notin M$ (convergence in the ℓ^∞-norm). Therefore M is not closed.

(d) [15p] Show that

$$\overline{M} = \{a \in \ell^\infty(\mathbb{N}) : a_j \to 0 \text{ as } j \to \infty\}.$$

To prove "$\supseteq$": Let $x \in \{a \in \ell^\infty(\mathbb{N}) : a_j \to 0 \text{ as } j \to \infty\}$ and define $x^n := (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$. Clearly $x^n \in M$ for all n and clearly $\|x^n - x\|_\infty \to 0$.

Conversely, to prove "$\subseteq$": Suppose that $x^n \in M$ for all n and suppose that $x^n \to x$ as $n \to \infty$ in $\ell^\infty(\mathbb{N})$. We must prove that $x_j \to 0$ as $j \to \infty$.

Let $\varepsilon > 0$. Then $\exists N$ such that $\|x^n - x\|_\infty < \varepsilon$ for all $n > N$. So $\left|x_j^n - x_j\right| < \varepsilon$ for all $n > N$ and for all j.

Now since $x^n \in M$, we know that $x_j^n = 0$ for all sufficiently large j. Hence $|x_j| < \varepsilon$ for all sufficiently large j and we are done.

(e) [15p] Show that $\exists$ a bounded linear functional $l \in \ell^\infty(\mathbb{N})^*$ such that

(a) $l(m) = 0$ for all $m \in M$; and

(b) $l(1,1,1,1,1,1,1,1,1,1,\ldots) = 1$.

By Corollary 4.12, there exists a linear functional l such that (i) $l = 0$ on M; $l(1,1,1,1,\ldots) = \operatorname{dist}((1,1,1,1,\ldots), M)$; and (iii) $\|l\| = 1$.

Since

$$\|(1,1,1,1,1,\ldots) - m\|_\infty \geq 1$$

for all $m \in M$ and since

$$\|(1,1,1,1,1,\ldots) - (1,0,0,0,0,\ldots)\|_\infty = 1,$$

we are finished.

Soru 3 (Weak Convergence)**.** Let X be a Banach space.

(a) [10p] Let (x_n) be a sequence in X. Give the definition of x_n *converges weakly* to x (i.e. $x_n \rightharpoonup x$ as $n \to \infty$).

> We say that x_n *converges weakly* to x (and write $x_n \rightharpoonup x$ as $n \to \infty$) iff $l(x_n) \to l(x)$ as $n \to \infty$, for all $l \in X^*$.

(b) [10p] Show that the weak limit is unique (i.e. show that if $x_n \rightharpoonup x$ and $x_n \rightharpoonup \tilde{x}$, then $x = \tilde{x}$).

> Suppose that $x_n \rightharpoonup x$ and $x_n \rightharpoonup \tilde{x}$. Then
>
> $$l(\tilde{x} - x) = l(\tilde{x}) - l(x) = \lim_{n\to\infty} l(x_n) - \lim_{n\to\infty} l(x_n) = 0$$
>
> for all $l \in X^*$. It follows that $\tilde{x} - x = 0$. Hence weak limits are unique.

Now suppose that

- X is a Hilbert space;
- $Y \subseteq X$;
- $\operatorname{span} Y$ is dense in X;
- (x_n) is a bounded sequence in X;
- $x \in X$;
- for any $y \in Y$, $\lim_{n\to\infty} \langle x_n - x, y \rangle = 0$.

(c) [30p] Show that $x_n \rightharpoonup x$ as $n \to \infty$.

> We are told that $\lim_{n\to\infty} \langle x_n - x, y \rangle = 0$ for all $y \in Y$. By linearity, we can show (you fill in the details) that $\lim_{n\to\infty} \langle x_n - x, y \rangle = 0$ for all $y \in \operatorname{span} Y$.
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> Let $\varepsilon > 0$ and let $h \in X$. Let $C = \sup_n \{\|x\|, \|x_n\|, 1\}$. Clearly $C < \infty$ since (x_n) is bounded.
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> Since $\operatorname{span} Y$ is dense in X, we can choose $y \in \operatorname{span} Y$ such that $\|h - y\| < \frac{\varepsilon}{4C}$. Since $\lim_{n\to\infty} \langle x_n - x, y \rangle = 0$, we can assume that $|\langle x_n - x, y \rangle| < \frac{\varepsilon}{2}$ for sufficiently large n.
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> Then for sufficiently large n, we have that
>
> $$\begin{aligned} |\langle x_n - x, h \rangle| &= |\langle x_n - x, y \rangle + \langle x_n - x, h - y \rangle| \\ &\leq |\langle x_n - x, y \rangle| + |\langle x_n - x, h - y \rangle| \\ &\leq |\langle x_n - x, y \rangle| + \|x_n - x\| \, \|h - y\| \\ &\leq \frac{\varepsilon}{2} + 2C\frac{\varepsilon}{4C} \\ &= \varepsilon \end{aligned}$$
>
> by the Cauchy-Schwarz Inequality. Hence $\langle x_n, h \rangle \to \langle x, h \rangle$.
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> It follows that $x_n \rightharpoonup x$ by the Riesz Representation Theorem.