2016.05.18 **MAT462 Fonksiyonel Analiz II – Final Sınavın Çözümleri** N. Course

Soru 1 (Hilbert-Schmidt Operators). Let X be a Hilbert space.

(a) [1p] Please write your student number on every page.

(b) [3p] Give the definition of the *Hilbert-Schmidt norm* of an operator.

$$\|K\|_2 := \left(\sum_j s_j(K)^2 \right)^{\frac{1}{2}}$$

where $\{s_j(K)\}$ are the singular values of $K : X \to X$.

(c) [3p] Give the definition of $\mathcal{J}_2(X)$, the space of *Hilbert-Schmidt operators*.

$$\mathcal{J}_2(X) := \{K \in \mathcal{K}(X) : \|K\|_2 < \infty\}.$$

Now consider the Hilbert space $\ell^2(\mathbb{N})$. Let $a = (a_j)_{j=1}^{\infty} \subseteq \mathbb{C}$ be a sequence. Define an operator $A : \ell^2(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ by

$$Ax_j = a_j x_j.$$

[For example: If $a = (3, 1, 4, 1, 5, 9, 2, \ldots)$ then $A(x_1, x_2, x_3, x_4, \ldots) = (3x_1, x_2, 4x_3, x_4, \ldots)$. If $a = (1, 0, 0, 0, 0, \ldots)$ then $A(x_1, x_2, x_3, x_4, \ldots) = (x_1, 0, 0, 0, \ldots)$.]

(d) [1p] Does $\{\delta^n\}_{n=1}^{\infty}$ form an orthonormal basis for $\ell^2(\mathbb{N})$?

☑ Yes ☐ No.

(e) [17p] Show that

$$a \in \ell^2(\mathbb{N}) \qquad \Longleftrightarrow \qquad A \text{ is a Hilbert-Schmidt operator.}$$

Since $\{\delta^n\}$ forms an orthonormal basis of $\ell^2(\mathbb{N})$ (part (d) was a hint!!!!), we can use Lemma 5.5 to calculate that

$$\|A\|_2 = \left(\sum_{j=1}^{\infty} \left\| A\delta^j \right\|^2 \right)^{\frac{1}{2}} = \left(\sum_j |a|^2 \right)^{\frac{1}{2}} = \|a\|_{\ell^2}.$$

Clearly if A is Hilbert-Schmidt, then $\|a\|_{\ell^2} = \|A\|_2 < \infty$ and hence that $a \in \ell^2(\mathbb{N})$. Conversely, suppose that $a \in \ell^2(\mathbb{N})$. Then we have that $\|A\|_2 = \|a\|_{\ell^2} < \infty$ and, by Lemma 5.6, this is sufficient to prove that A is a Hilbert-Schmidt operator.

Soru 2 (Finite Rank Operators).

(a) [3p] Give the definition of the *rank* of an operator.

Let $A : X \to Y$ be an operator. Then the rank of A is

$$\operatorname{rank}(A) := \dim A(X) = \dim \operatorname{Ran}(A).$$

(b) [2p] Give the definition of a *finite rank* operator.

An operator A is called a finite rank operator if and only if $\operatorname{rank}(A) < \infty$.

Define an operator $B : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ by

$$B(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) := (x_2, x_4, x_2, x_4, x_2, x_4, \ldots).$$

(c) [5p] Calculate $\operatorname{rank}(B)$.

An easy question: Clearly

$$\operatorname{Ran}(B) = \operatorname{span}\{(1,0,1,0,1,0,1,0,\ldots),(0,1,0,1,0,1,0,1,\ldots)\}.$$

Therefore $\operatorname{rank}(B) = 2$.

Now suppose that

- X is a Hilbert space;
- $A : X \to X$ is a finite rank operator; and
- $A^* : X \to X$ is the adjoint of A.

(d) [15p] Show that A^* is a finite rank operator.

SOLUTION 1: Since A is finite rank, we can write

$$A = \sum_{j=1}^{n} s_j \langle u_j, \cdot\rangle v_j$$

where u_j and v_j are as in Theorem 5.1 and where $n = \operatorname{rank}(A) \in \mathbb{N}$. Since

$$\begin{aligned}\langle A^* f, g\rangle = \langle f, Ag\rangle &= \left\langle f, \sum_{j=1}^{n} s_j \langle u_j, g\rangle v_j \right\rangle = \sum_{j=1}^{n} s_j \langle u_j, g\rangle \langle f, v_j\rangle \\ &= \left\langle \sum_{j=1}^{n} s_j \overline{\langle f, v_j\rangle} u_j, g \right\rangle = \left\langle \sum_{j=1}^{n} s_j \langle v_j, f\rangle u_j, g \right\rangle\end{aligned}$$

for all f and g, we can see that

$$A^* = \sum_{j=1}^{n} s_j \langle v_j, \cdot\rangle u_j.$$

Therefore $\operatorname{rank}(A^*) = n < \infty$.

SOLUTION 2: In MAT461 Fonksiyonel Analiz I, you showed that $\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^\perp$. Hence $\operatorname{Ker}(A) = \operatorname{Ran}(A^*)^\perp$ and thus $\operatorname{Ker}(A)^\perp = \operatorname{Ran}(A^*)$. It is easy to see that $A|_{\operatorname{Ker}(A)^\perp} \operatorname{Ker}(A)^\perp \to \operatorname{Ran}(A)$ is an isomorphism. Hence $\operatorname{Ran}(A^*) = \operatorname{Ker}(A)^\perp \cong \operatorname{Ran}(A)$. Therefore $\operatorname{rank}(A^*) = \operatorname{rank}(A) < \infty$.

Soru 3 (The Fredholm Alternative). Let X be a Hilbert space.

(a) [5p] Give the definition of the *orthogonal complement* of a set $M \subseteq X$.

The *orthogonal complement* of M is

$$M^{\perp} := \{f \in X : \langle f, g\rangle = 0 \ \forall g \in M\}.$$

Recall Theorem 5.10. You may assume that this theorem is true.

Theorem 10. *Suppose $K \in \mathcal{K}(X)$. Then*

$$\dim \operatorname{Ker}(1-K) = \dim \operatorname{Ran}(1-K)^{\perp} < \infty.$$

(b) [20p] Suppose that $K \in \mathcal{K}(X)$. Show that either

- the inhomogeneous equation
$$f = Kf + g$$
has a unique solution for every $g \in X$;

or

- the corresponding homogeneous equation
$$f = Kf$$
has a nontrivial solution.

(This famous result is called the *Fredholm Alternative*. It is named after the Swedish mathematician *Erik Ivar Fredholm*, 1866-1927.)

This is an easy question – I hope that you choose this question.

We know that either

(a) $\operatorname{Ker}(1-K) = \{0\}$; or

(b) $\operatorname{Ker}(1-K) \neq \{0\}$

CASE 1: Suppose that $\operatorname{Ker}(1-K) = \{0\}$. Then $\dim \operatorname{Ran}(1-K)^{\perp} = \dim \operatorname{Ker}(1-K) = 0$ which implies that $\operatorname{Ran}(1-K) = X$. But this means that $\forall g \in X$, $\exists f \in X$ such that $(1-K)f = g$. Because $\operatorname{Ker}(1-K) = \{0\}$, the f must be unique.
Therefore $f = Kf + g$ has a unique solution $\forall g \in X$.

CASE 2: Now suppose that $\operatorname{Ker}(1-K) \neq \{0\}$. Because K is linear, we know that $\dim \operatorname{Ker}(1-K) \geq 1$. Therefore $\exists f \neq 0$ such that $(1-K)f = 0$. Hence the equation $f = Kf$ has a nontrivial solution.

Soru 4 (Weak Convergence). Let X be a Banach space.

(a) [5p] Give the definition of *weak convergence* of a sequence of vectors in X.

We say that x_n converges weakly to x, and write $x_n \rightharpoonup x$, iff $l(x_n) \to l(x)$ for all $l \in X^*$.

Now suppose that

- $l_j \in X^*$ for all $j \in J$;
- $\{l_j\}_{j\in J}$ is total in X^*.

(b) [5p] Show that

$$x_n \rightharpoonup x \qquad \Longrightarrow \qquad x_n \text{ is bounded and } l_j(x_n) \to l_j(x) \text{ for all } j.$$

Another easy question for you: By definition if $x \rightharpoonup x$ then $l_j(x_n) \to l_j(x)$ for all j. 2

The boundedness of x_n comes immediately from Lemma 4.16 – every weak convergent sequence is weak Cauchy, and every weak Cauchy sequence is bounded. 3

It is also true that

$$x_n \rightharpoonup x \qquad \Longleftarrow \qquad x_n \text{ is bounded and } l_j(x_n) \to l_j(x) \text{ for all } j,$$

but I am not asking you to prove this. Perhaps this is a hint for part (c).

(c) [15p] Show that

$$x_n \rightharpoonup x \qquad \not\Leftarrow \qquad l_j(x_n) \to l_j(x) \text{ for all } j.$$

[HINT: Maybe consider $X = \ell^2(\mathbb{N})$ for your counterexample? You know that $\ell^2(\mathbb{N})^* \cong \ell^2(\mathbb{N})$.]

Define $l_j(z) = z_j$ for all $z \in \ell^2(\mathbb{N})$. Then $\{l_j\}$ is total in $\ell^2(\mathbb{N})^* \cong \ell^2(\mathbb{N})$. Moreover, define a linear functional l by $l(z) = \sum_{j=1}^{\infty} \frac{1}{j} z_j = \langle (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots), z \rangle$. Note that l is bounded because $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \in \ell^2(\mathbb{N})$.

Consider the sequence $x_n = n\delta^n = (0, \ldots, 0, n, 0, \ldots)$. Clearly $l_j(x^n) = 0$ for all $n > j$. Therefore $l_j(x^n) \to 0 = l_j(0)$ for all j. However x^n is not bounded and $l(x^n) = 1 \not\to 0$ which shows that $x \not\rightharpoonup 0$.

Soru 5 (Odds and Sods).

(a) [3p] Give the definition of the *graph* of an operator.

Let $A : \mathfrak{D}(A) \subseteq X \to Y$ be an operator. The graph of A is

$$\Gamma(A) = \{(x, Ax) : x \in \mathfrak{D}(A)\}.$$

(b) [2p] Give the definition of a *closed* operator.

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closed iff, its graph is a closed subset of $X \oplus Y$.

(c) [8p] Give an example of a closed operator. Prove that your operator is closed.

Many possible answers. E.g. identity operator on a Banach space.

(d) [5p] Give the definition of a *reflexive* space.

Define $J : X \to X^{**}$ by $J(x)(l) = l(x)$. X is called a *reflexive* space if and only if J is a surjection.

(e) [7p] Show that every Hilbert space is reflexive.

And another easy one to finish with: This comes immediately from the Riesz Representation Theorem. Shame on you if you can't get these very easy points.

By Riesz, $X \cong X^*$ for every Hilbert space. Hence if X is a Hilbert space, then $X \cong X^* \cong X^{**}$ and thus X must be reflexive.